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Gravity, spinors and gauge-natural bundles

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ABSTRACT

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GRAVITY, SPINORS AND GAUGE-NATURAL BUNDLES

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The purpose of this thesis is to give a fully gauge-natural formulation of gravitation theory, which turns out to be essential for a correct geometrical formulation of the coupling between gravity and spinor fields. In Chapter 1 we recall the necessary background material from differential geometry and introduce the fundamental notion of a gauge-natural bundle. Chapter 2 is devoted to expounding the general theory of Lie derivatives, its specialization to the gauge-natural context and, in particular, to spinor structures. In Chapter 3 we describe the geometric approach to the calculus of variations and the theory of conserved quantities. Then, in Chapter 4 we give our gauge-natural formulation of the Einstein (-Cartan) -Dirac theory and, on applying the formalism developed in the previous chapter, derive a new gravitational superpotential, which exhibits an unexpected freedom of a functorial origin. Finally, in Chapter 5 we complete the picture by presenting the Hamiltonian counterpart of the Lagrangian formalism developed in Chapter 3, and proposing a multisymplectic derivation of bi-instantaneous dynamics.

Appendices supplement the core of the thesis by providing the reader with useful background information, which would nevertheless disrupt the main development of the work. Appendix A is devoted to a concise account of categories and functors. In Appendix B we review some fundamental notions on vector fields and flows, and prove a simple, but useful, proposition. In Appendix C we collect the basic results that we need on Lie groups, Lie algebras and Lie group actions on manifolds. Finally, Appendix D consists of a short introduction to Clifford algebras and spinors.

Ai miei genitori

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Introduction

> Vos igitur, doctrinae et sapientiae filii, perquirite in hoc libro colligendo nostram dispersam intentionem quam in diversis locis proposuimus et quod occultatum est a nobis in uno loco, manifestus fecimus illud in alio, ut sapientibus vobis patefiat.
>
> H. C. Agrippa von Nettesheim, *De occulta philosophia*, III, lxv

It is commonly accepted nowadays that the appropriate mathematical arena for classical field theory is that of fibre bundles or, more precisely, of their jet prolongations (*cf.*, e.g., Atiyah 1979; Trautman 1980; Saunders 1989; Giachetta *et al.* 1997). What is less often realized or stressed is that, in physics, fibre bundles are always considered *together with some special class of morphisms*, i.e. as elements of a particular *category*. In other words, fields are always considered together with a particular class of *transformations*.

The category of *natural bundles* was introduced about thirty years ago and proved to be an extremely fruitful concept in differential geometry. But it was not until recently, when a suitable generalization was introduced, that of *gauge-natural bundles*, that the relevance of this functorial approach to physical applications began to be clearly perceived. The notion of naturality traditionally relates to the idea of coordinate invariance, or "covariance". The more recent introduction of gauge invariance into physics gives rise to the very idea of gauge-natural bundles.

Indeed, *every* classical field theory can be regarded as taking place on some jet prolongation of some gauge-natural (vector or affine) bundle associated with some principal bundle over a given base manifold (Eck 1981; Kolář *et al.* 1993; Fatibene 1999).

On the other hand, it is well known that one of the most powerful tools of Lagrangian field theory is the so-called "Noether theorem". It turns out that, when phrased in modern geometrical terms, this theorem crucially involves the concept of a Lie derivative, and here is where the aforementioned functorial approach is not only useful, but also *intrinsically unavoidable*. By using the general theory of Lie derivatives, one can see that the concept of Lie differentiation is, crucially, a *category-dependent* one, and it makes a real difference in taking the Lie derivative of, say, a vector field if one regards the tangent bundle as a purely natural bundle or, alternatively, as a more general gauge-natural bundle associated with some suitable principal bundle (*cf.* Chapter 2).

In Chapter 4 we shall show that this functorial approach is *essential* for a correct geometrical formulation of the Einstein (-Cartan) -Dirac theory and, at the same time, yields an unexpected *freedom* in the concept of conserved quantities. As we shall see, this freedom originates from the very fact that, when coupled with Dirac fields, Einstein's general relativity can no longer be regarded as a purely natural theory, because, in order to incorporate spinors, one *must* enlarge the class of morphisms of the theory.

This is the general idea which underlies the present work. An interesting by-product of this analysis is the successful systematization of the long-debated concept of a Lie derivative of spinor fields. A synopsis of the thesis follows.

In Chapter 1 we recall the necessary background material from differential geometry and introduce the fundamental notion of a gauge-natural bundle. Chapter 2 is devoted to expounding the general theory of Lie derivatives, its specialization to the gauge-natural context and, in particular, to spinor structures. In Chapter 3 we describe the geometric approach to the calculus of variations and the theory of conserved quantities. Then, in Chapter 4 we give our gauge-natural formulation of the Einstein (-Cartan) -Dirac theory and, on applying the formalism developed in the previous chapter, derive a new gravitational superpotential. Finally, in Chapter 5 we complete the picture by presenting the Hamiltonian counterpart of the Lagrangian formalism developed in Chapter 3, and proposing a multisymplectic derivation of “bi-instantaneous” dynamics, i.e. dynamics with two evolution directions.

Appendices supplement the core of the thesis by providing the reader with useful background information, which would nevertheless disrupt the main development of the work. Appendix A is devoted to a concise account of categories and functors. In Appendix B we review some fundamental notions on vector fields and flows, and prove a simple, but useful, proposition. In Appendix C we collect the basic results that we need on Lie groups, Lie algebras and Lie group actions on manifolds. Finally, Appendix D consists of a short introduction to Clifford algebras and spinors.

Most of the material contained in Chapters 2 and 4 is original and is partly based on Godina & Matteucci (2002), Godina *et al.* (2000, 2001) and Matteucci (2002). The rest of the thesis consists of the author’s original reformulation of standard material, often supplemented by explicit calculations which would be difficult to find elsewhere in the literature. Furthermore, §5.4 is original.

Chapter 1

Background differential geometry

Ubi materia, ibi geometria.
J. KEPLER, *De fundamentis astrologiae certioribus*, XX, 26

Ἴσμεν που ὅτι τῷ ὅλῳ καὶ παντὶ διοίσει ἡμμένος τε γεωμετρίας καὶ μή.
PLATO, *Res publica*, VII, ix, 527c

In this chapter we shall give a concise outline of some non-trivial concepts of differential geometry used throughout the thesis. In particular, we shall introduce the fundamental notion of a gauge-natural bundle.

1.1 Preliminaries and notation

Throughout the thesis, all maps are assumed to be of class C^∞, while manifolds are real, finite-dimensional, Hausdorff, second-countable and, hence, paracompact, unless otherwise stated. The adjectives "differentiable" and "smooth" are regarded as synonyms of C^∞ and used interchangeably.

The canonical pairing between 1-forms and vector fields will be denoted by $\langle \cdot, \cdot \rangle$, and the interior product of a vector field with a p-form by $\rfloor$ (*cf.*, e.g., Abraham *et al.* 1989).

The space of vector fields on a manifold M is denoted by $\mathfrak{X}(M)$, the space of (differential) p-forms on M by $\Omega^p(M)$. The tangent [cotangent] space at a point $x \in M$ is denoted by T_xM [T^*_xM]. The tangent [cotangent] bundle of M is denoted by TM [T^*M]. The *tangent map* induced by a smooth map $\varphi\colon M \to N$ between manifolds is defined to be the linear mapping $T_x\varphi\colon T_xM \to T_{\varphi(x)}N$ such that $\big(T_x\varphi(v)\big)(f) = v(f \circ \varphi)$ for all $x \in M$, $v \in T_xM$, $f \in C^\infty(N;\mathbb{R})$ (*cf.* Appendix B). We shall denote by $T\varphi\colon TM \to TN$ the total mapping given by $T\varphi|_{T_xM} = T_x\varphi$. If $\varphi\colon M \to N$ is a *diffeomorphism*, i.e. if it is smooth and has a smooth inverse $\varphi^{-1}\colon N \to M$, then we can define the induced *cotangent map*[1] $T^*_x\varphi := ((T_x\varphi)^{-1})^*\colon T^*_xM \to T^*_{\varphi(x)}N$ and the corresponding total mapping $T^*\varphi\colon T^*M \to T^*N$. In this case, we can also define the *push-forward* $\varphi_*\xi \in \mathfrak{X}(N)$ of a vector field $\xi \in \mathfrak{X}(M)$ by φ as $\varphi_*\xi := T\varphi \circ \xi \circ \varphi^{-1}$, whereas, in order to define the

[1]Note that in many texts the cotangent map is defined to be the *inverse* of this map. Our definition, though, is the more useful one in a functorial context such as the one of this thesis.

pull-back $\varphi^*\omega \in \Omega^p(M)$ of a p-form $\omega \in \Omega^p(N)$ by φ, $\varphi\colon M \to N$ need only be a smooth map. Indeed, for $\underset{i}{v} \in T_xM$, this is given by

$$(\varphi^*\omega)_x(\underset{1}{v}, \ldots, \underset{p}{v}) = \omega_{\varphi(x)}(T_x\varphi \circ \underset{1}{v}, \ldots, T_x\varphi \circ \underset{p}{v}).$$

Lower-case Greek indices are assumed to be (holonomic) coordinate indices on M and run from 0 to $\dim M - 1$. Lower-case Latin indices are assumed to be anholonomic indices on $T_xM \cong \mathbb{R}^{\dim M}$ and run from 0 to $\dim M - 1$. Lower-case Gothic indices are assumed to label the fibre coordinates of a generic fibre bundle B over M (*cf.* §1.2). Finally, upper-case calligraphic letters denote Lie algebra indices running from 1 to the dimension of the algebra, whereas primed and unprimed upper-case italic letters denote 2-spinor indices running from $0'$ to $1'$ and from 0 to 1, respectively.

Let $(\partial_\mu := \partial/\partial x^\mu|_x)$ be the natural basis of T_xM corresponding to a set of local coordinates (x^λ) around a point $x \in M$, and $(\mathrm{d}x^\mu := \mathrm{d}x^\mu|_x)$ its dual basis. Define

$$\mathrm{d}s_{\mu_1\ldots\mu_k} := \partial_{\mu_k} \lrcorner\, (\cdots \lrcorner\, (\partial_{\mu_1} \lrcorner\, \mathrm{d}s)\cdots), \tag{1.1.1}$$

where $k = 0, 1, \ldots, m-1$, $m := \dim M$, and $\mathrm{d}s := \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{m-1}$. The set $(\mathrm{d}s_{\mu_1\ldots\mu_k})$ forms a basis of the vector space $\bigwedge^{m-k}T^*_xM$. On exploiting the formal properties of the interior product and the fact that $\Omega^p(M) = \{0\}$ for $p > m$, one easily finds that

$$\mathrm{d}x^\rho \wedge \mathrm{d}s_\mu = \delta^\rho{}_\mu\, \mathrm{d}s, \tag{1.1.2a}$$

$$\mathrm{d}x^\rho \wedge \mathrm{d}s_{\mu\nu} = (\delta^\rho{}_\nu\, \mathrm{d}s_\mu - \delta^\rho{}_\mu\, \mathrm{d}s_\nu), \tag{1.1.2b}$$

$$\mathrm{d}x^\rho \wedge \mathrm{d}s_{\mu\nu\sigma} = (\delta^\rho{}_\mu\, \mathrm{d}s_{\nu\sigma} - \delta^\rho{}_\nu\, \mathrm{d}s_{\mu\sigma} + \delta^\rho{}_\sigma\, \mathrm{d}s_{\mu\nu}). \tag{1.1.2c}$$

Furthermore, one has

$$\mathrm{d}s_{\mu_1\ldots\mu_k} = \frac{1}{(m-k)!} e_{\mu_1\ldots\mu_k\nu_{k+1}\ldots\nu_m}\, \mathrm{d}x^{\nu_{k+1}} \wedge \cdots \wedge \mathrm{d}x^{\nu_m}, \tag{1.1.1$'$}$$

where $e_{\mu_1\ldots\mu_m} := m!\,\delta^0{}_{[\mu_1} \cdots \delta^{m-1}{}_{\mu_m]}$. Throughout, we use Bourbaki's (1967) convention on the exterior product, whereby $\alpha \wedge \beta \equiv \alpha \otimes \beta - \beta \otimes \alpha$ for any two 1-forms α and β on M.

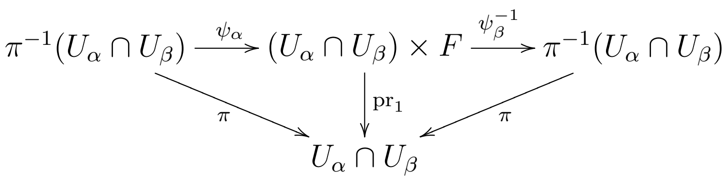
1.2 Fibre bundles

Let M be an m-dimensional manifold and $(x^\lambda)_{\lambda=0}^{m-1}$ a set of coordinates on a chart (U, φ) of M, U being an open subset of M and $\varphi\colon U \to \mathbb{R}^m$ a local homeomorphism. In the sequel, we shall often denote any such chart simply by (U, x^λ), or even just (x^λ) if there is no danger of confusion. Moreover, we shall denote by the same symbol x^λ both the map $\varphi(U) \to \mathbb{R}$ and the composite map $U \to \varphi(U) \to \mathbb{R}$, as customary.

A *fibred manifold* is a triple (B, M, π), where B and M are two manifolds (called the *bundle* or *total space* and the *base*, respectively) and $\pi\colon B \to M$ is a differentiable, surjective map of constant rank $r = \dim M$, called the *projection*. The preimage $B_x := \pi^{-1}(x)$ of a point $x \in M$ is a submanifold of B, called the *fibre* over x. A *fibred chart* $(V, x^\lambda, y^\mathfrak{a})$ of B is a chart such that $(\pi(V), x^\lambda)$ is a chart of M. A *fibred atlas* is an atlas of fibred charts; its *transition functions* read[2]

$$x'^{\lambda} = \varphi^{\lambda}(x^{\mu}),$$
$$y'^{\mathfrak{a}} = \Phi^{\mathfrak{a}}(x^{\mu}, y^{\mathfrak{b}}).$$

Definition 1.2.1. We call a quadruple $(B, M, \pi; F)$ a (***differentiable***) ***fibre bundle*** over M if (B, M, π) is a fibred manifold and F is a manifold, called the *standard* (or *typical*) *fibre*, such that for any $x \in M$ there exist a chart (U, φ) of M with $x \in U$ and a diffeomorphism $\psi\colon \pi^{-1}(U) \to U \times F$ such that $\mathrm{pr}_1 \circ \psi = \pi$, pr_1 denoting the projection of $U \times F$ onto the first factor, i.e. onto U. Furthermore, if two charts of M $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) have a non-empty intersection, then the following diagram is required to be commutative



ψ_α and ψ_β being the diffeomorphisms associated with U_α and U_β, respectively. The pairs $\{(U_\alpha, \psi_\alpha)\}$, or simply the maps $\{\psi_\alpha\}$, are called the (*local*) *trivializations* of the bundle. Every fibre B_x is naturally diffeomorphic to the standard fibre F. The maps

$$\psi_{\alpha\beta} := \psi_\alpha \circ \psi_\beta^{-1}\colon (U_\alpha \cap U_\beta) \times F \to (U_\alpha \cap U_\beta) \times F$$

are called the *transition functions* of the bundle; for any $x \in U_\alpha \cap U_\beta$ they define a diffeomorphism $\tilde{\psi}_{\alpha\beta}(x)$ of F into itself by

$$(\tilde{\psi}_{\alpha\beta}(x))(f) = \psi_{\alpha\beta}(x, f), \quad f \in F.$$

The dimension of B, of course, turns out to be the sum of the dimensions of M and F. When no confusion can arise, a fibre bundle $(B, M, \pi; F)$ will be denoted simply by (B, M, π) or even B, just as the underlying fibred manifold or its total space, respectively. A fibre bundle of the form $(M \times F, M, \mathrm{pr}_1; F)$ is called a *trivial bundle*.

Let now G be a Lie group, i.e. a group which is also a (differentiable) manifold and where the composition and inversion maps are differentiable[3]. Let G act on the standard fibre F of the bundle $(B, M, \pi; F)$ in such a way that, for any transition function $\psi_{\alpha\beta}$,

$$\psi_{\alpha\beta}(x, f) = (x, a_{\alpha\beta}(x) \cdot f), \tag{1.2.1}$$

$a_{\alpha\beta}(x)$ being an element of G and '$\cdot$' denoting the (left) action of G on F. Then we say that $(B, M, \pi; F; G)$ is a ***fibre bundle with structure group*** G or, for short, a G***-bundle***. The maps $a_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$ are called the *transition functions with values*

[2] Here and in the sequel a notation like $f(z^\mu)$ is to be understood as a shorthand for $f((z^\mu))$.

[3] For a concise introduction to Lie groups and Lie algebras see Appendix C.

in G and enjoy the following properties:

$$\begin{gathered} a_{\beta\alpha}(x) = a_{\alpha\beta}(x)^{-1}, \\ a_{\alpha\beta}(x)a_{\beta\gamma}(x) = a_{\alpha\gamma}(x) \end{gathered}$$

for any $x \in U_\alpha \cap U_\beta \cap U_\gamma \neq \varnothing$, juxtaposition denoting group multiplication.

A G-bundle $(B, M, \pi; F; G)$ is called:

- a ***principal (fibre) bundle***, if $F \equiv G$, and the action of G on itself is simply the group multiplication (on the left);
- a ***vector bundle***, if F is a vector space $\mathcal{V}$, and G is isomorphic to a subgroup of the general linear group $\mathrm{GL}(\mathcal{V})$, acting on $\mathcal{V}$ in the standard way;
- an ***affine bundle*** (***modelled on a vector bundle*** E [over M]), if F is an affine space $\mathcal{A}$ modelled on the standard fibre of E, and G is isomorphic to a subgroup of the general affine group $\mathrm{GA}(\mathcal{A})$, acting on $\mathcal{A}$ in the standard way.

For instance, the tangent bundle TM of an m-dimensional manifold M can be regarded as a vector bundle over M with standard fibre $\mathbb{R}^m$ and structure group $\mathrm{GL}(m, \mathbb{R})$ since the transition functions are linear on every fibre.

Let (B, M, π) be a fibred manifold and $\varphi\colon M' \to M$ a smooth map between manifolds. By the *pull-back* of B by φ we shall mean the fibred manifold $(\varphi^*B, M', \mathrm{pr}_1)$ with total space $\varphi^*B := \{ (x', y) \in M' \times B \mid \pi(y) = \varphi(x') \}$. If (B', M, π') is another fibred manifold over the same base M, then we can also define the *fibred product* $(B \times_M B', M, \tilde{\pi})$ of (B, M, π) and (B', M, π') as the fibred manifold $(\pi^*B, M, \pi \circ \mathrm{pr}_1)$. If (B, M, π) and (B', M, π') are actually vector bundles, we shall usually write $B \oplus_M B'$ for $B \times_M B'$, which is consistent since $(B \oplus_M B')_x \cong B_x \oplus B'_x$ for all $x \in M$.

Now, let (B, M, π) and (B', M', π') be two fibred manifolds; a *fibred (manifold) morphism* Φ (*over* φ) is the pair (φ, Φ), where $\varphi \in C^\infty(M, M')$, $\Phi \in C^\infty(B, B')$ and $\pi' \circ \Phi = \varphi \circ \pi$. Equivalently, the following diagram is commutative.

$$\begin{array}{ccc} B & \xrightarrow{\Phi} & B' \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle\pi'} \\ M & \xrightarrow[\varphi]{} & M' \end{array}$$

A *fibred isomorphism* is a fibred morphism where both φ and Φ are diffeomorphisms. A *fibred automorphism* is a fibred isomorphism where $(B', M', \pi') \equiv (B, M, \pi)$. A *base-preserving (fibred) morphism* is a fibred morphism where $M' \equiv M$ and $\varphi \equiv \mathrm{id}_M$. If a fibred manifold has an additional structure (of a vector, principal bundle, etc.), its morphisms are defined in such a way that the fibre structure is preserved: so, for instance, a "vector bundle morphism" is linear when restricted to the fibres.

A (*local*) *section* σ of a fibred manifold (B, M, π) is a differentiable map from $U \subseteq M$ to B such that $\pi \circ \sigma = \mathrm{id}_U$. A *global section* is a section defined on the whole of M. All fibre bundles possess local sections, but not all bundles possess global sections, e.g.

a principal bundle admits a global section iff it is trivial[4], whereas (smooth) vector and affine bundles possess an infinite number of global sections provided the dimension of their standard fibres is greater than one[5]. In particular, a vector bundle admits the so-called *zero section* $0\colon M \to B$ given by $0(x) = 0_x \in B_x$ for all $x \in M$.

The ***vertical*** (***tangent***) ***bundle*** (VB, B, ν_B) of a fibred manifold (B, M, π) is defined to be the submanifold $\ker(T\pi) \subseteq TB$ such that $V_yB = \ker(T_y\pi)$ for any $y \in B$. The elements of the fibre (at y)

$$V_yB \equiv \{\, v \in T_yB \mid T_y\pi(v) = 0 \in T_{\pi(y)}M \,\}$$

are called *vertical vectors* (at y) and can be regarded as vectors tangent to B which are also tangent to the fibres of (B, M, π). A ***vertical vector field*** is a differentiable map $\Upsilon\colon M \to VB$ such that $\pi \circ \nu_B \circ \Upsilon = \mathrm{id}_M$. Locally,

$$\Upsilon(x, y) \equiv \Upsilon^{\mathfrak{a}}(x^\lambda, y^{\mathfrak{b}}) \frac{\partial}{\partial y^{\mathfrak{a}}}\bigg|_y ,$$

$(x^\lambda, y^{\mathfrak{a}})$ being local fibred coordinates around a point $\psi_\alpha^{-1}(x, y) \in B$.

Vertical vector fields are just a particular instance of a larger class of vector fields, called "projectable vector fields". A ***projectable vector field*** $\Xi \in \mathfrak{X}(B)$ over a vector field $\xi \in \mathfrak{X}(M)$ is a vector field on B such that

$$T\pi \circ \Xi = \xi \circ \pi$$

or, locally,

$$\Xi^\mu(x^\lambda, y^{\mathfrak{a}}) = \xi^\mu(x^\lambda),$$

whenever $\xi = \xi^\mu \partial_\mu|_x$ and $\Xi = \Xi^\mu \partial_\mu|_x + \Xi^{\mathfrak{a}} \partial_{\mathfrak{a}}|_y$, $(x^\lambda, y^{\mathfrak{a}})$ being local fibred coordinates around a point $\psi_\alpha^{-1}(x, y) \in B$. (In the sequel, we shall tend to omit the points at which natural basis vectors are evaluated.)

1.3 More on principal bundles

In the sequel, we shall denote a principal bundle $(P, M, \pi; G; G)$ simply by $(P, M, \pi; G)$, or even $P(M, G)$ whenever we do not need to specify the projection π.

Any principal bundle $P(M, G)$ admits a (canonical) ***right action*** $\tilde{R}\colon P \times G \to P$ or, equivalently, $\tilde{R}_a \equiv \tilde{R}(\cdot, a)\colon P \to P$ locally given by

$$(\tilde{R}_a)_\alpha\colon \psi_\alpha^{-1}(x, b) \mapsto \psi_\alpha^{-1}(x, R_a b) \equiv \psi_\alpha^{-1}(x, ba),$$

[4]Indeed, for any trivialization (U_α, ψ_α) of a principal bundle $(P, M, \pi; G; G)$, we can define the local section $\sigma_\alpha\colon x \in U_\alpha \mapsto \sigma_\alpha(x) := \psi_\alpha^{-1}(x, e)$, where e denotes the identity of G. Then, if ψ_α is global (and hence the bundle is trivial), so is σ_α. Conversely, if σ_α is global, so is $\psi_\alpha\colon \sigma_\alpha(x)\cdot a \mapsto \psi_\alpha(\sigma_\alpha(x)\cdot a) \equiv (x, a)$, where a is an element of G and '$\cdot$' denotes the right action of G on P (see the next section).

[5]Indeed, if $\{(U_\alpha, \psi_\alpha)\}$ is a vector or affine bundle atlas, any smooth mapping φ_α from U_α to the standard fibre defines a local section $x \mapsto \psi_\alpha^{-1}(x, \varphi_\alpha(x))$ on U_α. Then, if $\{f_\alpha\}$ is a partition of unity subordinated to $\{U_\alpha\}$, a global section is readily given by $x \mapsto \sum_\alpha f_\alpha(x)\psi_\alpha^{-1}(x, \varphi_\alpha(x))$.

which turns out to be independent of the trivialization used. Indeed, if $\{\psi_\beta\}$ is another trivialization of $P(M,G)$, then certainly

$$\psi_\alpha^{-1}(x,b) = \psi_\beta^{-1}(x, a_{\beta\alpha}(x)b),$$

for some transition function $a_{\beta\alpha}$, whence the diagram

$$\begin{array}{ccc} \psi_\alpha^{-1}(x,b) & \xrightarrow{(\tilde{R}_a)_\alpha} & \psi_\alpha^{-1}(x,ba) \\ \| & & \| \\ \psi_\beta^{-1}(x,a_{\beta\alpha}(x)b) & \xrightarrow[(\tilde{R}_a)_\beta]{} & \psi_\beta^{-1}(x,a_{\beta\alpha}(x)ba) \end{array}$$

is commutative, and globality follows. If $u \in P$, we shall often simply write $u \cdot a$ for $\tilde{R}_a u$. The right action is obviously *vertical*, i.e. $u \cdot a \in P_x$ for all $u \in P_x$ and $a \in G$, it is *free*, i.e. $u \cdot a = u$, $u \in P$, implies $a = e$, e denoting the unit element of G, and is *transitive* on the fibres, i.e., if $u, u' \in P_x$, then $u' = u \cdot a$ for some $a \in G$.

A *homomorphism* of a principal bundle $(P, M, \pi; G)$ into another principal bundle $(P', M', \pi'; G')$ consists of a differentiable mapping $\Phi\colon P \to P'$ and a Lie group homomorphism $f\colon G \to G'$ such that $\Phi(u \cdot a) = \Phi(u) \cdot f(a)$ for all $u \in P$, $a \in G$. Hence, Φ maps fibres into fibres and induces[6] a differentiable mapping $\varphi\colon M \to M'$ by $\varphi(x) = \pi'(\Phi(u))$, u being any point in P such that $\pi(u) = x$. A homomorphism $\Phi\colon P \to P'$ is called an *embedding* if $\varphi\colon M \to M'$ is an embedding[7] and $f\colon G \to G'$ is injective. In such a case, we can identify P with $\Phi(P)$, G with $f(G)$ and M with $\varphi(M)$, and P is said to be a *subbundle* of P'. If $M' = M$ and $\varphi = \mathrm{id}_M$, P is called a *reduced subbundle* or a *reduction* of P', and we also say that G' "reduces" to the subgroup G.

A homomorphism $\Phi\colon P \to P'$ is called an *isomorphism* if there exists a homomorphism of principal bundles $\Psi\colon P' \to P$ such that $\Psi \circ \Phi = \mathrm{id}_P$ and $\Phi \circ \Psi = \mathrm{id}_{P'}$. An isomorphism $\Phi\colon P \to P$ is called an *automorphism*.
In the sequel, we shall need only a restricted class of principal bundle morphisms.

Definition 1.3.1. Let $P(M,G)$ and $P'(M',G)$ be two principal G-bundles. A ***principal morphism*** from P to P' is a principal bundle homomorphism $\Phi\colon P \to P'$ such that $f = \mathrm{id}_G$.

The class of all principal G-bundles over m-dimensional manifolds together with the class of all principal morphisms between any two of them forms a category in the sense of Definition A.1.18, which we shall denote by $\boldsymbol{PB}_m(G)$. In particular, we have

Definition 1.3.2. Let $P(M,G)$ be a principal bundle. A (***principal***) ***automorphism*** of P is a G*-equivariant* diffeomorphism of P onto itself, i.e. a diffeomorphism $\Phi\colon P \to P$ such that $\Phi(u \cdot a) = \Phi(u) \cdot a$ for all $u \in P, a \in G$.

We shall denote by $\mathrm{Aut}(P)$ the group of all automorphisms of P.

[6] One can convince oneself that this is true by following almost exactly the same argument as the one used in the proof of Proposition 1.3.4 below.

[7] We recall that a mapping $\varphi\colon M \to M'$ is an *embedding* if it is injective, and $T_x\varphi\colon T_xM \to T_{\varphi(x)}M'$ is injective for all $x \in M$.

Definition 1.3.3. Let Ξ be a vector field on P generating the one-parameter group $\{\Phi_t\}$. Then, Ξ is called a (***right***) ***G-invariant vector field*** if Φ_t is an automorphism of P for all $t \in \mathbb{R}$.

Now, if Ξ is a G-invariant vector field generating the one-parameter group $\{\Phi_t\}$ on P, differentiating the expression $\Phi_t(u) \cdot a = \Phi_t(u \cdot a)$ with respect to t at $t = 0$ yields

$$\left(T\tilde{R}_a \circ \Xi\right)(u) = \left(\Xi \circ \tilde{R}_a\right)(u)$$

for all $u \in P$ and $a \in G$, or, equivalently,

$$(\tilde{R}_a)_* \Xi = \Xi$$

for all $a \in G$, i.e. Ξ is indeed "(right) G-invariant" (*cf.* §C.2).
We shall denote by $\mathfrak{X}_G(P)$ the Lie algebra of G-invariant vector fields of P.

Proposition 1.3.4. *Let $(P, M, \pi; G)$ be a principal bundle. Each automorphism $\Phi \in \mathrm{Aut}(P)$ induces a unique diffeomorphism $\varphi\colon M \to M$ such that $\pi \circ \Phi = \varphi \circ \pi$.*

Proof. Since the right action of G on P is transitive on the fibres, if $u, u' \in P_x$, then $u' = u \cdot a$ for some $a \in G$. Therefore, for such u and u'

$$\pi(\Phi(u')) = \pi(\Phi(u \cdot a)) = \pi(\Phi(u) \cdot a) = \pi(\Phi(u)),$$

where the second equality follows from the fact that Φ is a principal automorphism, and the third one from the transitivity of the right action. Therefore, if we define $\varphi(x)$ to equal $\pi(\Phi(u'))$, this is independent of the choice of u'. It remains to show that φ is a diffeomorphism. So, let σ be any (local) section of P defined in a neighbourhood U of x. Then, $\varphi|_U = \pi \circ \Phi \circ \sigma$, demonstrating that φ is smooth at x: analogously one shows that φ^{-1} is smooth. Finally, φ is unique because π is surjective. □

Locally, we can aways express an automorphism Φ of P as

$$\Phi(x, a) = (\varphi(x), f(x)a) \tag{1.3.1}$$

for all $(x, a) \in U \times G$, $U \subset M$, where $\varphi\colon M \to M$ is the unique diffeomorphism such that $\pi \circ \Phi = \varphi \circ \pi$ as per the previous proposition, and $f\colon U \to G$ is a local map. Indeed, if $\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is a local trivialization of $P(M, G)$, then, owing to Proposition 1.3.4 (and the fact that G is a group), there must exist some $c \in G$ such that

$$\Phi(\psi_\alpha^{-1}(x, a)) = \psi_\alpha^{-1}(\varphi(x), ac) \tag{1.3.2}$$

for all $\psi_\alpha^{-1}(x, a) \in P$. Furthermore, there certainly exists a local map $\tilde{f}\colon U_\alpha \times G \to G$ such that $\tilde{f}(x, a) = c$. Now, from the definition of a right action and that of a principal automorphism it follows that

$$\Phi(\psi_\alpha^{-1}(x, ab)) = \Phi(\psi_\alpha^{-1}(x, a) \cdot b) = \Phi(\psi_\alpha^{-1}(x, a)) \cdot b$$

for all $\psi_\alpha^{-1}(x, a) \in P$ and $b \in G$. Using (1.3.2) and the definition of $\tilde{R}_b$ once again, the previous identity can be rewritten as

$$\psi_\alpha^{-1}(\varphi(x), ab\tilde{f}(x,ab)) = \psi_\alpha^{-1}(\varphi(x), a\tilde{f}(x,a)b).$$

This, in turn, implies that

$$b\tilde{f}(x,ab) = \tilde{f}(x,a)b$$

or, multiplying on the left by b^{-1},

$$\tilde{f}(x,ab) = b^{-1}\tilde{f}(x,a)b,$$

whence, setting $a = e$,

$$\tilde{f}(x,b) = b^{-1}f(x)b,$$

where $f\colon U_\alpha \to G$ is a local map such that $f(x) = \tilde{f}(x,e)$. But then, substituting $\tilde{f}(x,a) = a^{-1}f(x)a$ for c into (1.3.2), we recover precisely (1.3.1). □

Corollary 1.3.5. *Let $P(M,G)$ be a principal bundle. Then, every G-invariant vector field Ξ on P is projectable over a unique vector field ξ on the base manifold M.*

Proof. From Proposition 1.3.4, if $\{\Phi_t\}$ denotes the flow of Ξ, then there is a unique diffeomorphism $\varphi_t\colon M \to M$ such that $\pi \circ \Phi_t = \varphi_t \circ \pi$. If we differentiate this expression with respect to t at $t = 0$ and define $\xi \in \mathfrak{X}(M)$ as $\frac{\partial}{\partial t}\varphi_t\Big|_{t=0}$, then we get

$$T\pi \circ \Xi = \xi \circ \pi,$$

and realize that ξ is the required vector field. □

Every G-invariant vector field Ξ on P admits the following local representation:

$$\Xi(x,a) = \xi^\mu(x)\partial_\mu + \Xi^{\mathcal{A}}(x)\rho_{\mathcal{A}}(a) \tag{1.3.3}$$

for all $(x,a) \in U \times G$, $U \subset M$, where $\xi(x) =: \xi^\mu(x)\partial_\mu$ and $(\rho_{\mathcal{A}})$ is a basis of right-invariant vector fields on G given by (*cf.* §C.2)

$$\rho_{\mathcal{A}}(a) = T_e R_a \varepsilon_{\mathcal{A}}$$

at any $a \in G$, $(\varepsilon_{\mathcal{A}})$ being a basis of $T_e G \cong \mathfrak{g}$. Indeed, in accordance with (1.3.1) and taking Corollary 1.3.5 into account, the flow of Ξ must be locally expressible as

$$\Phi_t(x,a) = (\varphi_t(x), f_t(x)a) \equiv (\varphi_t(x), R_a f_t(x)).$$

Differentiating this expression with respect to t at $t = 0$, we get

$$\Xi(x,a) = \xi(x) + T_e R_a \Xi_e(x), \tag{1.3.4}$$

where

$$\Xi_e(x) = \frac{\partial}{\partial t} f_t(x)\Big|_{t=0} \in T_e G \cong \mathfrak{g}.$$

Hence, on writing $\Xi_e(x) =: \Xi^{\mathcal{A}}(x)\varepsilon_{\mathcal{A}}$, we recover precisely (1.3.3). □

In the sequel, we shall need the transformation rule for a G-invariant vector field Ξ under a change of local trivialization, i.e. the transformation rule for (1.3.3) or, equivalently, (1.3.4). Now, in accordance with (1.2.1), we can express any change of trivialization on P by some transition function $\psi_{\alpha\beta}\colon U_\alpha \cap U_\beta \times G \to U_\alpha \cap U_\beta \times G$ satisfying

$$\psi_{\alpha\beta}(x,a) = (x, a_{\alpha\beta}(x)a) \equiv (x, R_a a_{\alpha\beta}(x)) \equiv (x, L_{a_{\alpha\beta}(x)}a) \tag{1.3.5}$$

for all $x \in U_\alpha \cap U_\beta$, $a \in G$, and some $a_{\alpha\beta}\colon U_\alpha \cap U_\beta \to G$. Accordingly, the trivialization change on TP induced by (1.3.5) will be represented by the transition function

$$T_{(x,a)}\psi_{\alpha\beta}\colon T_x(U_\alpha \cap U_\beta) \oplus T_a G \to T_x(U_\alpha \cap U_\beta) \oplus T_a G.$$

Then, all that is left to do is to evaluate $T_{(x,a)}\psi_{\alpha\beta}$ on $\xi(x) + T_e R_a \Xi_e(x)$, which is:

$$T_{(x,a)}\psi_{\alpha\beta}(\xi(x) + T_e R_a \Xi_e(x)) = \xi(x) + T_x(R_a \circ a_{\alpha\beta}) \circ \xi(x) + T_a L_{a_{\alpha\beta}(x)} \circ T_e R_a \Xi_e(x).$$

Now, the second term on the r.h.s. of this expression can be obviously rewritten as

$$\begin{aligned} T_x(R_a \circ a_{\alpha\beta}) \circ \xi(x) &= T_{a_{\alpha\beta}(x)} R_a \circ T_x a_{\alpha\beta} \circ \xi(x) \\ &= T_e R_{a_{\alpha\beta}(x)a} \circ T_{a_{\alpha\beta}(x)} R_{a_{\alpha\beta}(x)^{-1}} \circ T_x a_{\alpha\beta} \circ \xi(x), \end{aligned}$$

whereas for third one we have:

$$\begin{aligned} T_a L_{a_{\alpha\beta}(x)} \circ T_e R_a \Xi_e(x) &= T_e(L_{a_{\alpha\beta}(x)} \circ R_a) \circ \Xi_e(x) \\ &= T_e(R_a \circ R_{a_{\alpha\beta}(x)} \circ R_{a_{\alpha\beta}(x)^{-1}} \circ L_{a_{\alpha\beta}(x)}) \circ \Xi_e(x) \\ &= T_e R_{a_{\alpha\beta}(x)a} \circ \mathrm{Ad}_{a_{\alpha\beta}(x)} \Xi_e(x), \end{aligned}$$

Ad denoting the adjoint representation of G (*cf.* §C.2). Thus, we are finally left with

$$\begin{aligned} T_{(x,a)}\psi_{\alpha\beta}(\xi(x) + T_e R_a \Xi_e(x)) &= \xi(x) \\ &+ T_e R_{a_{\alpha\beta}(x)a} \circ (T_{a_{\alpha\beta}(x)} R_{a_{\alpha\beta}(x)^{-1}} \circ T_x a_{\alpha\beta} \circ \xi(x) + \mathrm{Ad}_{a_{\alpha\beta}(x)} \Xi_e(x)). \end{aligned} \tag{1.3.6}$$

Specializing this result to the situation when $\xi = 0$, we obtain

$$T_{(x,a)}\psi_{\alpha\beta}(T_e R_a \Upsilon_e(x)) = T_e R_{a_{\alpha\beta}(x)a} \circ \mathrm{Ad}_{a_{\alpha\beta}(x)} \Upsilon_e(x) \tag{1.3.7}$$

as the transformation rule for any *vertical* G-invariant vector field Υ on P. Making use of (1.3.3), replacing a with b and then setting $a(x) := a_{\alpha\beta}(x)$, we can rewrite (1.3.6) and (1.3.7) in a slightly more evocative form, as

$$\Xi(x, a(x)b) = \xi^\mu(x)\partial_\mu + (T_{a(x)} R_{a(x)^{-1}} \circ \xi^\mu(x)\partial_\mu a(x) + \mathrm{Ad}_{a(x)} \Xi_e(x))^{\mathcal{A}} \rho_{\mathcal{A}}(a(x)b) \tag{1.3.8}$$

and

$$\Upsilon(x, a(x)b) = (\mathrm{Ad}_{a(x)} \Upsilon_e(x))^{\mathcal{A}} \rho_{\mathcal{A}}(a(x)b), \tag{1.3.9}$$

respectively.

1.3.1 Bundle of linear frames and G-structures

Definition 1.3.6. Let M be an m-dimensional manifold and $\mathcal{B}_x$ the set of all bases of $T_xM \cong \mathbb{R}^m$, $x \in M$. Let LM denote the disjoint union $\coprod_{x\in M} \mathcal{B}_x$. Since $\mathcal{B}_x$ is isomorphic to $\mathrm{GL}(m,\mathbb{R})$ for all $x \in M$, $LM(M, \mathrm{GL}(m,\mathbb{R}))$ is a principal bundle known as the ***bundle of linear frames*** over M.

If (U, x^λ) is a chart on M, a local trivialization of LM over U is given by the chart $(\pi^{-1}(U); x^\lambda, u^\mu{}_a = e_a{}^\mu)$, where the coordinates $(e_a{}^\mu)$ are the components, relative to the basis (∂_μ), of the vectors (e_a) forming a basis of T_xM, i.e.

$$e_a \equiv e_a{}^\mu \, \partial_\mu.$$

Definition 1.3.7. Let G be a Lie subgroup of $\mathrm{GL}(m,\mathbb{R})$. By a ***G-structure*** on M we shall mean a subbundle $P(M,G)$ of LM.

According to the previous definition, the principal bundles (over M) LM, $\mathrm{CSO}(M,g)$ and $\mathrm{SO}(M,g)$ with structure groups $\mathrm{GL}(m,\mathbb{R})$, $\mathrm{CSO}(p,q)^e \equiv \mathrm{SO}(p,q)^e \times \mathbb{R}^+$ and $\mathrm{SO}(p,q)^e$, respectively, where g is an otherwise unspecified metric tensor on M of signature (p,q) and $p+q=m$ (*cf.* §C.1), are all examples of G-structures on M.

Now, in accordance with (1.3.3), a G-invariant vector field on a G-structure $P(M,G)$ will be locally written as

$$\Xi = \xi^\mu \partial_\mu + \Xi^a{}_b \rho_a{}^b, \tag{1.3.10}$$

where $(\rho_a{}^b)$ is a basis of G-invariant vector fields in their *fundamental representation* on $\mathfrak{gl}(m,\mathbb{R})$, i.e. in their lowest dimensional faithful (linear) representation on $\mathfrak{gl}(m,\mathbb{R})$. If $(x^\mu, u^a{}_b)$ denote local fibred coordinates around a point $\psi_\alpha^{-1}(x,a) \in P$, then

$$\rho_a{}^b(a) \equiv u^b{}_c \circ a \left.\frac{\partial}{\partial u^a{}_c}\right|_a .$$

Indeed, in these coordinates, the natural basis of $T_eG \cong \mathfrak{g}$ reads $(\partial/\partial u^a{}_b|_e)$, whence

$$\begin{aligned}
\rho_a{}^b(a) &\equiv T_eR_a \left.\tfrac{\partial}{\partial u^a{}_b}\right|_e \\
&= T_eR_a \delta^c{}_a \delta^b{}_d \left.\tfrac{\partial}{\partial u^c{}_d}\right|_e \\
&= \delta^c{}_a \delta^b{}_e a^e{}_d \left.\tfrac{\partial}{\partial u^c{}_d}\right|_a = a^b{}_c \left.\tfrac{\partial}{\partial u^a{}_c}\right|_a ,
\end{aligned}$$

having set $a^b{}_c := u^b{}_c \circ a$. □

Analogously, a vertical G-invariant vector field on a G-structure $P(M,G)$ will be locally expressible as

$$\Upsilon = \Upsilon^a{}_b \rho_a{}^b. \tag{1.3.11}$$

Hence, transformation rules (1.3.8) and (1.3.9) specialized to (1.3.10) and (1.3.11) read

$$\Xi'^a{}_b = \xi^\mu \partial_\mu a^a{}_c \tilde{a}^c{}_b + a^a{}_c \Xi^c{}_d \tilde{a}^d{}_b, \tag{1.3.12}$$

$\tilde{a}^a{}_b := (a^{-1})^a{}_b$, and

$$\Upsilon'^a{}_b = a^a{}_c \Upsilon^c{}_d \tilde{a}^d{}_b, \tag{1.3.13}$$

respectively. In particular, on LM one can choose $a^a{}_\mu = \theta^a{}_\mu$ corresponding to the change of basis $\partial_\mu \mapsto e_a \equiv e_a{}^\mu \partial_\mu$, $\|e_a{}^\mu\| := \|\theta^a{}_\mu\|^{-1}$. Then (1.3.12) becomes

$$\Xi^a{}_b = e_b{}^\nu \xi^\mu \partial_\mu \theta^a{}_\nu + \theta^a{}_\mu \Xi^\mu{}_\nu e_b{}^\nu \tag{1.3.14}$$

and, correspondingly,

$$\Xi^\mu{}_\nu = \theta^a{}_\nu \xi^\mu \partial_\mu e_a{}^\nu + e_a{}^\mu \Xi^a{}_b \theta^b{}_\nu. \tag{1.3.14$'$}$$

Remark 1.3.8. Note that, in general, (1.3.14) and (1.3.14′) do not make sense on any G-structure other than LM. This is because the $e_a{}^\mu$'s *cannot* be regarded as local coordinates on any *proper* Lie subgroup of $\mathrm{GL}(m, \mathbb{R})$. In Chapter 2, though, we shall introduce the important concept of a G-tetrad, thanks to which we shall be able to regard (1.3.14) and (1.3.14′) as the transformations between a $\mathrm{GL}(m, \mathbb{R})$-invariant vector field on LM and the corresponding (G-invariant) Kosmann vector field on a given G-structure $P(M, G)$ (provided G is a reductive Lie subgroup of G: *cf.* Corollary 2.4.14 and Definition 2.3.1 below).

1.4 Associated bundles

Let $P(M, G)$ be a principal bundle and $\lambda\colon G \times F \to F$ a left action of G on a manifold F. The ***associated*** (***fibre***) ***bundle*** $P \times_\lambda F$, or simply $(P \times F)/G$, is the G-bundle over M with standard fibre F whose total space is the quotient of $P \times F$ with respect to the equivalence relation

$$(u', f') \sim (u, f) \iff \exists a \in G \mid u' = u \cdot a \text{ and } f' = a^{-1} \cdot f := \lambda(a^{-1}, f), \tag{1.4.1}$$

which is clearly a right action of G on $P \times F$. The equivalence classes will be denoted by $[u, f]_\lambda$. Indeed, if $\psi_\alpha\colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ is a local trivialization of $P(M, G)$, we can define a local trivialization $(\psi_\lambda)_\alpha\colon \pi_\lambda^{-1}(U_\alpha) \to U_\alpha \times F$ of $P \times_\lambda F$ as

$$(\psi_\lambda)_\alpha^{-1}(x, f) := [\psi_\alpha^{-1}(x, e), f]_\lambda$$

for all $(x, f) \in U_\alpha \times F$, e denoting the unit element of G. The map $q\colon P \times F \to P \times_\lambda F : (u, f) \mapsto [u, f]_\lambda$ is known as the *quotient map*.

As an example, consider the following left action of $\mathrm{GL}(m, \mathbb{R})$ on $\mathbb{R}^m$:

$$\begin{cases} \lambda\colon \mathrm{GL}(m, \mathbb{R}) \times \mathbb{R}^m \to \mathbb{R}^m \\ \lambda\colon (\alpha^c{}_b, v^a) \mapsto v'^a := \alpha^a{}_b v^b \end{cases}. \tag{1.4.2}$$

If we choose a set of adapted coordinates $(x^\lambda, u^\mu{}_a = e_a{}^\mu)$ on LM, a corresponding set of coordinates on the associated bundle $LM \times_\lambda \mathbb{R}^m$ is given by the quotient map

$$\begin{cases} q\colon LM \times \mathbb{R}^m \to LM \times_\lambda \mathbb{R}^m \\ q\colon (x^\lambda, e_a{}^\mu, v^b) \mapsto (x^\lambda, y^\mu := e_b{}^\mu v^b) \end{cases}. \tag{1.4.3}$$

It is then immediate to realize that $LM \times_\lambda \mathbb{R}^m \cong TM$, i.e. that $LM \times_\lambda \mathbb{R}^m$ is (canonically) isomorphic to the tangent bundle of M: indeed, (1.4.2) and (1.4.3) simply state that a point in $LM \times_\lambda \mathbb{R}^m$ is an object which transforms like a vector in the classical sense. Analogously, one can show that the cotangent bundle and, more generally, any tensor (density) bundle over M may be regarded as a (vector) bundle associated with LM.

Note, though, that, if we have a metric tensor g on M of signature (p, q), we could also consider the action

$$\begin{cases} \lambda' \colon \mathrm{SO}(p,q)^e \times \mathbb{R}^m \to \mathbb{R}^m \\ \lambda' \colon (a^c{}_b, v^a) \mapsto v'^a := a^a{}_b v^b \end{cases}, \tag{1.4.4}$$

and the associated fibre bundle $\mathrm{SO}(M,g) \times_{\lambda'} \mathbb{R}^m$ would be still isomorphic to TM. Hence we see that the tangent bundle and, more generally, any tensor (density) bundle over M could be equally well regarded as a (vector) bundle associated with $\mathrm{SO}(M,g)$.

1.5 Principal connections

Definition 1.5.1. Let $P(M,G)$ be a principal bundle. A ***principal connection*** on P is a fibre G-equivariant projection $\varkappa\colon TP \to VP$ such that $\varkappa \circ \varkappa = \varkappa$ and $\operatorname{im} \varkappa = VP$. Here, "$G$-equivariant" means that $T\tilde{R}_a \circ \varkappa = \varkappa \circ T\tilde{R}_a$ for all $a \in G$.

It follows that $HP := \ker \varkappa$ is a constant-rank vector subbundle of TP, called the *horizontal bundle*. We have a decomposition $TP = HP \oplus VP$ and $T_uP = H_uP \oplus V_uP$ for all $u \in P$. Then, $\varkappa$ is also called the *vertical projection* and the projection $\chi := \mathrm{id}_{TP} - \varkappa$, which is also G-equivariant and satisfies $\chi \circ \chi = \chi$ and $\operatorname{im} \chi = \ker \varkappa$, is called the *horizontal projection*.

Equivalently, $\varkappa$ can be viewed as 1-form in $\Omega^1(P, TP)$, which, owing to its G-equivariance, locally must be of the form

$$\varkappa = (\pi^{\mathcal{A}} + \omega^{\mathcal{A}}{}_\mu(x)\, \mathrm{d}x^\mu) \otimes \rho_{\mathcal{A}}, \tag{1.5.1}$$

$(\pi^{\mathcal{A}})$ denoting the basis of (right) G-invariant forms on G dual to $(\rho_{\mathcal{A}})$. Analogously, χ must locally read

$$\chi = \mathrm{d}x^\mu \otimes (\partial_\mu - \omega^{\mathcal{A}}{}_\mu(x)\rho_{\mathcal{A}}). \tag{1.5.2}$$

Given a G-invariant vector field Ξ on P locally given by (1.3.3), we can then define its *horizontal* and *vertical parts* as

$$\hat{\Xi} := \Xi \lrcorner \chi$$

and

$$\check{\Xi} := \Xi \lrcorner \varkappa \equiv \Xi - \Xi \lrcorner \chi,$$

respectively. Locally,

$$\hat{\Xi} = \xi^\mu \partial_\mu - \omega^{\mathcal{A}}{}_\mu \xi^\mu \rho_{\mathcal{A}}, \tag{1.5.3}$$

$$\check{\Xi} = (\Xi^{\mathcal{A}} + \omega^{\mathcal{A}}{}_\mu \xi^\mu) \rho_{\mathcal{A}}. \tag{1.5.4}$$

Remark 1.5.2. Note that, although the general definition of a vertical vector field on P is independent of the choice of a connection (*cf.* §1.2), both the horizontal *and* the vertical part of a G-invariant vector field on P depend on the particular connection chosen. In this context, a vertical (G-invariant) vector field on P could be defined as a (G-invariant) vector field (on P) which coincides with its vertical part for *any* connection $\varkappa$ on P.

Regarding χ as 1-form on M, which we shall denote by the same letter, we can define the *horizontal lift* $\hat{\xi}$ of a vector field $\xi \in \mathfrak{X}(M)$ onto P as $\hat{\xi} := \xi \lrcorner \chi$. Locally,

$$\hat{\xi} = \xi^\mu \partial_\mu - \omega^{\mathcal{A}}{}_\mu \xi^\mu \rho_{\mathcal{A}} \equiv \hat{\Xi}. \tag{1.5.5}$$

Let now $P \times_\lambda F$ be a fibre bundle associated with P. We can define a horizontal lift $\hat{\xi}_\lambda$ of a vector field $\xi \in \mathfrak{X}(M)$ onto $P \times_\lambda F$ *induced* by $\hat{\xi}$ as

$$\hat{\xi}_\lambda([u, f]_\lambda) := T_{(u,f)}q(\hat{\xi}(u) + 0_F(f)), \tag{1.5.6}$$

$q\colon P \times F \to P \times_\lambda F$ denoting the quotient map, and 0_F the zero vector field on F. This horizontal lift implicitly defines a 1-form χ_λ on $P \times_\lambda F$ with values in $T(P \times_\lambda F)$ by $\xi \lrcorner \chi_\lambda := \hat{\xi}_\lambda$. Such a 1-form is known as the ***induced connection*** on $P \times_\lambda F$. Also, we can define the ***covariant derivative*** $\nabla_\xi \sigma$ of a section $\sigma\colon M \to P \times_\lambda F$ with respect to a vector field $\xi \in \mathfrak{X}(M)$ as

$$\nabla_\xi \sigma := T\sigma \circ \xi - \hat{\xi}_\lambda \circ \sigma. \tag{1.5.7}$$

Furthermore, if F is a (finite-dimensional) vector space $\mathcal{V}$ and $\lambda\colon G \to \mathrm{GL}(\mathcal{V})$ is a representation of G on $\mathcal{V}$, we can define the ***covariant exterior derivative*** $\mathcal{D}\alpha$ of a p-form α on M with values in the vector bundle $P \times_\lambda \mathcal{V}$ by the formula

$$\begin{aligned}\mathcal{D}\alpha(\underset{1}{\xi}, \ldots, \underset{p+1}{\xi}) &= \sum_{i=1}^{p} (-1)^{i+1} \nabla_{\underset{i}{\xi}} \alpha(\underset{1}{\xi}, \ldots, \widehat{\underset{i}{\xi}}, \ldots, \underset{p+1}{\xi}) \\ &\quad + \sum_{1 \leqslant i < j \leqslant p+1} (-1)^{i+j} \alpha([\underset{i}{\xi}, \underset{j}{\xi}], \underset{1}{\xi}, \ldots, \widehat{\underset{i}{\xi}}, \ldots, \widehat{\underset{j}{\xi}}, \ldots, \underset{p+1}{\xi}),\end{aligned} \tag{1.5.8}$$

$\underset{i}{\xi} \in \mathfrak{X}(M)$, a hat over a vector field denoting that it must be omitted. Of course, for a $(P \times_\lambda \mathcal{V})$-valued 0-form α on M, we have $\xi \lrcorner \mathcal{D}\alpha \equiv \mathcal{D}\alpha(\xi) = \nabla_\xi \alpha$.

Now, let $\mathfrak{g}$ denote the Lie algebra of G and let $(u, \xi_e) \in P \times \mathfrak{g}$. It is clear that the mapping $(u, \xi_e) \mapsto \big(T_{(u,e)}\tilde{R}\big)(0_u, \xi_e)$ defines a vector bundle isomorphism $P \times \mathfrak{g} \overset{\cong}{\to} VP$ over P, i.e. VP is trivial as a vector bundle over P. Therefore we have that $\omega(v) := (T_e \tilde{L}_u)^{-1} \varkappa(v)$ is in $\mathfrak{g}$ for all $v \in T_u P$, $\tilde{L}_u$ denoting the partial mapping $\tilde{R}(u, \cdot)\colon G \to P$. In this way, we get a $\mathfrak{g}$-valued 1-form $\omega \in \Omega^1(P; \mathfrak{g})$, known as the ***connection 1-form***.

Proposition 1.5.3. *If $\varkappa \in \Omega(P; VP)$ is a principal connection on a principal bundle $P(M, G)$, then the connection 1-form satisfies the following properties*:

(i) $\omega(T_e \tilde{L}_u \xi_e) = \xi_e$ *for all* $\xi_e \in \mathfrak{g}$;

(ii) $\big((\tilde{R}_a)^* \omega\big)(v) = \mathrm{Ad}_{a^{-1}} \omega(v)$ *for all* $a \in G$ *and* $v \in T_u P$.

Conversely, a 1-form $\omega \in \Omega^1(P; \mathfrak{g})$ satisfying (i) *defines a 1-form $\varkappa \in \Omega^1(P; VP)$ by $\varkappa(v) = T_e \tilde{L}_u \omega(v)$, which is a principal connection iff* (ii) *is satisfied.*

Proof. The proof of the proposition readily follows from direct computation.

(*i*) From the definition of ω and the verticality of $T_e\tilde{L}_u\xi$, one obtains:

$$T_e\tilde{L}_u\omega(T_e\tilde{L}_u\xi_e) = \varkappa(T_e\tilde{L}_u\xi_e) = T_e\tilde{L}_u\xi_e$$

for all $\xi_e \in \mathfrak{g}$. Since $T_e\tilde{L}_u \colon \mathfrak{g} \to V_uP$ is an isomorphism, the result follows.

(*ii*) First, note that (*cf.* §C.1)

$$(\tilde{R}_a \circ \tilde{L}_u)b = \tilde{R}_a u \cdot b = u \cdot aa^{-1}ba = u \cdot aI_{a^{-1}}b = (\tilde{L}_{u\cdot a} \circ I_{a^{-1}})b.$$

for all $a, b \in G$, $u \in P$. Hence (*cf.* §C.2),

$$(T_u\tilde{R}_a \circ T_e\tilde{L}_u)\xi_e = T_e(\tilde{R}_a \circ \tilde{L}_u)\xi_e = T_e(\tilde{L}_{u\cdot a} \circ I_{a^{-1}})\xi_e = T_e\tilde{L}_{u\cdot a} \circ \mathrm{Ad}_{a^{-1}}\xi_e \qquad (1.5.9)$$

for all $\xi_e \in \mathfrak{g}$. Now, from the definition of ω we have that

$$T_e\tilde{L}_{u\cdot a} \circ (\tilde{R}_a)^*\omega \circ v = T_e\tilde{L}_{u\cdot a}\omega(T_u\tilde{R}_a v) = \varkappa(T_u\tilde{R}_a v) \qquad (1.5.10a)$$

for all $v \in T_uP$. On the other hand, using (1.5.9), the definition of ω and the G-equivariance of $\varkappa$,

$$T_e\tilde{L}_{u\cdot a} \circ \mathrm{Ad}_{a^{-1}}\omega(v) = (T_u\tilde{R}_a \circ T_e\tilde{L}_u)\omega(v) = T_u\tilde{R}_a\varkappa(v) = \varkappa(T_u\tilde{R}_a v). \qquad (1.5.10b)$$

Now, from (1.5.10) and the fact that $T_e\tilde{L}_{u\cdot a} \colon \mathfrak{g} \to V_{u\cdot a}P$ is an isomorphism the result easily follows.

This completes the proof. □

Now, let $(P, M, \pi; G)$ be a principal bundle. Let $\{U_\alpha\}$ be an open covering of M with a family of trivializations $\psi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ and the corresponding family of transition functions $a_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$. For each α, let $\sigma_\alpha \colon U_\alpha \to P$ be the section on U_α defined by $\sigma_\alpha(x) = \psi_\alpha^{-1}(x, e)$, $x \in U_\alpha$, e denoting the identity of G. Let θ be the (left-invariant $\mathfrak{g}$-valued) *canonical* (or *Maurer-Cartan*) 1-*form on G* defined by

$$\theta(\xi_e) = \xi_e$$

for all $\xi_e \in T_eG \cong \mathfrak{g}$ or, equivalently,

$$\theta(v) = T_aL_{a^{-1}}v$$

for all $v \in T_aG$. For each non-empty intersection $U_\alpha \cap U_\beta$, define a $\mathfrak{g}$-valued 1-form on $U_\alpha \cap U_\beta$ as

$$\theta_{\alpha\beta} := a_{\alpha\beta}^*\theta,$$

and, for each α, define a $\mathfrak{g}$-valued 1-form ω_α on U_α as

$$\omega_\alpha := \sigma_\alpha^*\omega.$$

Then, we have the following

Proposition 1.5.4. *On $U_\alpha \cap U_\beta$, the forms $\theta_{\alpha\beta}$ and ω_α are related by the formula*

$$\omega_\beta = \mathrm{Ad}_{a_{\alpha\beta}^{-1}} \omega_\alpha + \theta_{\alpha\beta}. \tag{1.5.11}$$

Conversely, for every family of $\mathfrak{g}$-valued 1-forms $\{\omega_\alpha\}$ each defined on U_α and satisfying (1.5.11), *there is a* unique *connection 1-form ω which gives rise to $\{\omega_\alpha\}$ in the described manner.*

Proof. This is a classical proposition: we shall give Kobayashi & Nomizu's (1963) proof in a slightly modernized notation. If $U_\alpha \cap U_\beta$ is non-empty, then $\sigma_\beta(x) = \sigma_\alpha(x) \cdot a_{\alpha\beta}(x) \equiv \tilde{R}_{a_{\alpha\beta}(x)} \sigma_\alpha(x) \equiv \tilde{L}_{\sigma_\alpha(x)} a_{\alpha\beta}(x)$ for all $x \in U_\alpha \cap U_\beta$ since the right action is transitive on the fibres. Now,

$$\begin{aligned} T_x\sigma_\beta(v) &= T_x(\sigma_\alpha \cdot a_{\alpha\beta})(v) \\ &= T_{\sigma_\alpha(x)} \tilde{R}_{a_{\alpha\beta}(x)} \circ T_x \sigma_\alpha(v) + T_{a_{\alpha\beta}(x)} \tilde{L}_{\sigma_\alpha(x)} \circ T_x a_{\alpha\beta}(v) \end{aligned} \tag{1.5.12}$$

for all $v \in T_x(U_\alpha \cap U_\beta)$. Applying ω on both sides of (1.5.12) yields

$$\omega_\beta(v) = \mathrm{Ad}_{a_{\alpha\beta}(x)^{-1}} \omega_\alpha(v) + \theta_{\alpha\beta}(v), \tag{1.5.13}$$

which is clearly equivalent to (1.5.11). Indeed, as far as the l.h.s. of (1.5.13) is concerned, we obviously have

$$\omega\Big(T_x\sigma_\beta(v)\Big) \equiv \sigma_\beta^* \omega(v) \equiv \omega_\beta(v).$$

As for the first term on the r.h.s., it follows directly from Proposition 1.5.3(ii). Finally, let ξ be the left-invariant vector field on G which equals $T_x a_{\alpha\beta}(v)$ at $a := a_{\alpha\beta}(x)$ so that $\theta_{\alpha\beta}(v) \equiv \theta(T_x a_{\alpha\beta}(v)) = T_a L_{a^{-1}} \xi(a) \equiv \xi(e) \in \mathfrak{g}$. Now, set $u := \sigma_\alpha(x) \cdot a_{\alpha\beta}(x)$. With these substitutions the second term on the r.h.s. of (1.5.12) can be rewritten as

$$T_a \tilde{L}_{u \cdot a^{-1}} \xi(a) = T_a(\tilde{L}_u \circ L_{a^{-1}}) \xi(a) = T_e \tilde{L}_u \circ T_a L_{a^{-1}} \xi(a).$$

On applying ω to this expression and using Proposition 1.5.3(i), we find precisely $\theta_{\alpha\beta}(v)$, as required. This concludes the proof of the first part of the proposition.
The converse can be easily verified by following back the process of obtaining $\{\omega_\alpha\}$ from ω. □

Locally, setting $a := a_{\alpha\beta}$, we can rewrite transformation rule (1.5.11) as

$$\omega'^{\mathcal{A}}{}_\mu(x) = (\mathrm{Ad}_{a(x)^{-1}})^{\mathcal{A}}{}_{\mathcal{B}} \omega^{\mathcal{B}}{}_\mu(x) + (T_{a(x)} L_{a(x)^{-1}} \partial_\mu a(x))^{\mathcal{A}}, \tag{1.5.14}$$

where the $\omega^{\mathcal{A}}{}_\mu$'s are precisely the ones appearing in (1.5.1) and (1.5.2).

1.5.1 Linear connections

Definition 1.5.5. A principal connection on the bundle of linear frames LM is called a ***linear connection*** on M.

Now, if $P(M, G)$ is a G-structure on M, then, in the notation of §1.3.1, transformation rule (1.5.14) will read

$$\omega'^a{}_{b\mu} = \tilde{a}^a{}_c \partial_\mu a^c{}_b + \tilde{a}^a{}_c \omega^c{}_{d\mu} a^d{}_b.$$

In particular, on LM, if we make the choice $a^\mu{}_a = e_a{}^\mu$, then we have

$$\omega^a{}_{b\mu} = \theta^a{}_\nu \partial_\mu e_b{}^\nu + \theta^a{}_\rho \Gamma^\rho{}_{\nu\mu} e_b{}^\nu \tag{1.5.15}$$

as the transformation rule for a linear connection on M corresponding to a change from holonomic to anholonomic coordinates[8]. Similarly to (1.3.14) above, (1.5.15) does not make sense on any G-structure other than LM. Also in this case, though, after the introduction of the notion of G-tetrad in Chapter 2, it will be possible to regard (1.5.15) as the transformation rule between a principal connection on a G-structure and a linear connection on M (*cf.* Remark 1.3.8).

Clearly, in the case of a change of holonomic coordinates $(x^\lambda) \mapsto (x'^\lambda)$, (1.5.15) specializes to the well known transformation rule

$$\Gamma^\rho{}_{\nu\mu} \mapsto \Gamma'^\rho{}_{\nu\mu} = \frac{\partial x'^\rho}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial x'^\mu \partial x'^\nu} + \frac{\partial x'^\rho}{\partial x^\sigma} \frac{\partial x^\tau}{\partial x'^\nu} \frac{\partial x^\omega}{\partial x'^\mu} \Gamma^\sigma{}_{\tau\omega}. \tag{1.5.16}$$

We conclude by giving the local expressions for χ on LM, i.e.

$$\chi = \mathrm{d}x^\mu \otimes (\partial_\mu - \omega^a{}_{b\mu} \rho_a{}^b),$$

$\rho_a{}^b \equiv \theta^b{}_\mu \partial / \partial \theta^a{}_\mu$, or [*cf.* (1.5.15)]

$$\chi = \mathrm{d}x^\mu \otimes (\partial_\mu - \Gamma^\rho{}_{\nu\mu} \rho_\rho{}^\nu),$$

$\rho_\rho{}^\nu \equiv e_a{}^\nu \partial / \partial e_a{}^\rho$, and for χ_λ on $TM \cong LM \times_\lambda \mathbb{R}^m$, i.e.

$$\chi_\lambda = \mathrm{d}x^\mu \otimes (\partial_\mu - \omega^a{}_{b\mu} y^b e_a),$$

$e_a \equiv e_a{}^\mu \partial_\mu$, or

$$\chi_\lambda = \mathrm{d}x^\mu \otimes (\partial_\mu - \Gamma^\rho{}_{\nu\mu} y^\nu \partial_\rho),$$

$y^a \equiv \theta^a{}_\mu y^\mu$, obtained on using (1.5.6). Hence, on applying (1.5.7), we find

$$\nabla_\xi \eta = \xi^\mu (\partial_\mu \eta^a + \omega^a{}_{b\mu} \eta^b) e_a$$

or, equivalently,

$$\nabla_\xi \eta = \xi^\mu (\partial_\mu \eta^\rho + \Gamma^\rho{}_{\nu\mu} \eta^\nu) \partial_\rho,$$

$\eta^a \equiv \theta^a{}_\mu \eta^\mu$, as the local expression of the covariant derivative of a vector field $\eta \in \mathfrak{X}(M)$ with respect to another vector field $\xi \in \mathfrak{X}(M)$: this calculation can be easily generalized to any tensor (density) field on M. Note also that, if we define a *torsion tensor* τ on M as

$$\tau(\xi, \xi') = \nabla_\xi \xi' - \nabla_{\xi'} \xi - [\xi, \xi'], \tag{1.5.17}$$

we can implicitly define the *transpose* $\tilde{\chi}$ of the linear connection χ by means of the

[8] The use of a different kernel letter to denote a linear connection "in holonomic coordinates" is due in part to historical reasons, and in part to reasons which will become apparent in the sequel. Note also that in order to adhere to the conventions of most relativists, our $\Gamma^\rho{}_{\nu\mu}$ differs from Kobayashi & Nomizu's (1963) by the order of subscripts, and from Kolář *et al.*'s (1993) by a sign.

associated covariant derivative operator $\tilde{\nabla}$ on TM given by

$$\tilde{\nabla}_\xi \eta = \nabla_\xi \eta - \tau(\xi, \eta),$$

which is well defined, and locally reads

$$\tilde{\nabla}_\xi \eta = \xi^\mu (\partial_\mu \eta^a + \tilde{\omega}^a{}_{b\mu} \eta^b) e_a \tag{1.5.18}$$

or, equivalently,

$$\tilde{\nabla}_\xi \eta = \xi^\mu (\partial_\mu \eta^\rho + \tilde{\Gamma}^\rho{}_{\nu\mu} \eta^\nu) \partial_\rho,$$

where $\tilde{\omega}^a{}_{b\mu} := \theta^c{}_\mu e_b{}^\nu \omega^a{}_{c\nu}$ and $\tilde{\Gamma}^\rho{}_{\nu\mu} := \Gamma^\rho{}_{\mu\nu}$. The connection χ is then called *torsionless* or *symmetric* if $\tau \equiv 0$ or, equivalently, $\tilde{\chi} \equiv \chi$. In this case, of course, we also have $\tilde{\nabla} \equiv \nabla$ and $\Gamma^\rho{}_{\mu\nu} \equiv \Gamma^\rho{}_{\nu\mu}$. Finally, if (M, g) is a (pseudo-) Riemannian manifold (*cf.* §D.2), a linear connection χ on M is called the *Levi-Civita* (or *Riemannian*) *connection* if it is symmetric and *metric*, i.e.

$$\nabla g = 0, \tag{1.5.19}$$

'∇' denoting the associated covariant derivative operator on $\bigvee^2 T^*M$, and '$\bigvee$' the symmetrized tensor product. In this case, χ is unique.

1.6 Natural bundles

Many of the fibre bundles one normally uses in physics—such as the tangent, cotangent and, more generally, any tensor (density) bundle—are nearly always considered *together with a special class of morphisms*, according to which *all* fibred coordinate changes are induced by some coordinate changes on the base. E.g., in the case of the tangent bundle, equipped with fibred coordinates (x^λ, y^μ), once the base coordinate change $x^\lambda \mapsto x'^\lambda := \varphi^\lambda(x^\mu)$ is given, the fibred coordinate change

$$x^\lambda \mapsto x'^\lambda = \varphi^\lambda(x^\mu), \tag{1.6.1a}$$
$$y^\mu \mapsto y'^\mu = \Phi^\mu(x^\lambda, y^\nu) := J^\mu{}_\nu y^\nu \tag{1.6.1b}$$

is uniquely determined [here, $\|J^\mu{}_\nu := \partial\varphi^\lambda/\partial x^\mu\|$ denotes the Jacobian matrix of transformation (1.6.1*a*)]. Analogously, in the case of the cotangent bundle, equipped with fibred coordinates (x^λ, y_μ), for the same base coordinate change we have

$$x^\lambda \mapsto x'^\lambda = \varphi^\lambda(x^\mu), \tag{1.6.2a}$$
$$y_\mu \mapsto y'_\mu = \Phi_\mu(x^\lambda, y_\nu) := (J^{-1})_\mu{}^\nu y_\nu. \tag{1.6.2b}$$

Fibre bundles of this kind are called "bundles of geometric objects" or—when considered *together with* this special class of morphisms—"natural (fibre) bundles". In the latter *functorial* sense[9], they were first introduced by Nijenhuis (1972) (see also Salvioli 1972; Ferraris & Francaviglia 1983*a*; Kolář *et al.* 1993). Their precise definition follows.

[9]For a short account on categories and functors see Appendix A.

Definition 1.6.1. Let $\boldsymbol{FM}$ be the category of fibred manifolds and fibre-respecting morphisms, and $\boldsymbol{Mf}_m$ the category of m-dimensional manifolds and local diffeomorphisms. Let $\mathrm{Ob}(\cdot)$ denote the class of the objects of a given category. A ***natural bundle*** is a functor $\mathscr{F}\colon \boldsymbol{Mf}_m \to \boldsymbol{FM}$ such that:

(i) every manifold $M \in \mathrm{Ob}(\boldsymbol{Mf}_m)$ is transformed into a fibred manifold $(\mathscr{F}M, M, \pi) \in \mathrm{Ob}(\boldsymbol{FM})$;

(ii) every local diffeomorphism $\varphi\colon M \to M'$ between two manifolds $M, M' \in \mathrm{Ob}(\boldsymbol{Mf}_m)$ is transformed into a fibre-respecting morphism $\mathscr{F}\varphi\colon \mathscr{F}M \to \mathscr{F}M'$ over φ;

(iii) for every open subset $U \subseteq M$, $M \in \mathrm{Ob}(\boldsymbol{Mf}_m)$, $\mathscr{F}U = \pi^{-1}(U)$ and the inclusion $\iota\colon U \to M$ is transformed into the inclusion $\mathscr{F}\iota\colon \pi^{-1}(U) \to \mathscr{F}M$.

A section of $(\mathscr{F}M, M, \pi)$ is called a *natural object* or, sometimes, a *field of geometric objects*. If $(\mathscr{F}M, M, \pi)$ has an additional structure of a vector [affine] bundle and its morphisms $\{\mathscr{F}\varphi\}$ are vector [affine] bundle morphisms, then it is called *natural vector* [*affine*] *bundle*.

Remark 1.6.2. It should be clear from its very definition that a natural bundle $\mathscr{F}$ automatically induces a fibre bundle structure on the fibred manifold $(\mathscr{F}M, M, \pi)$, thereby justifying its name. Indeed, one can show that for any m-dimensional manifold M the quadruple $(\mathscr{F}M, M, \pi; \mathscr{F}_0\mathbb{R}^m)$ is a fibre bundle over M, $\mathscr{F}_0\mathbb{R}^m$ denoting the fibre of $\mathscr{F}\mathbb{R}^m$ over $0 \in \mathbb{R}^m$ (*cf.* Kolář *et al.* 1993, §14.2). In any case, in §1.10 we shall see an explicit construction of (gauge-) natural bundles as fibre bundles associated with a particular class of principal bundles (and morphisms thereon).

Remark 1.6.3. Nijenhuis's (1972) original definition of a natural bundle contained the following additional (regularity) condition:

(iv) if M, M' and M'' are three objects in $\boldsymbol{Mf}_m$ and $\varphi\colon M'' \times M \to M'$ is a smooth map such that for all $x \in M''$ the maps $\varphi_x := \varphi(x, \cdot)\colon M \to M'$ are local diffeomorphisms, then the map $\tilde{\mathscr{F}}\varphi\colon M'' \times \mathscr{F}M \to \mathscr{F}M'$ defined as $\tilde{\mathscr{F}}\varphi(x, \cdot) := \mathscr{F}\varphi_x$ is smooth.

Epstein & Thurston (1979) proved that this condition actually follows from the previous three.

From Definition 1.6.1 it follows immediately that the tangent [cotangent] bundle TM [T^*M] of an m-dimensional manifold M is a natural (vector) bundle (over M). Indeed, it is a functor

$$T\colon M \to TM \quad [T^*\colon M \to T^*M]$$

given by

(i) $T(U) := TU$ [$T^*(U) := T^*U$] for any open submanifold U of M;

(ii) $T(\varphi) := T\varphi$ [$T^*(\varphi) := T^*\varphi \equiv ((T\varphi)^{-1})^*$] for any local diffeomorphism φ of M.

Analogously, it is immediate to realize that any tensor (density) bundle is a natural (vector) bundle.

Remark 1.6.4. In the sequel, we shall often not distinguish between a natural bundle $\mathscr{F}$, the fibred manifold $(\mathscr{F}M, M, \pi)$, or even the total space $\mathscr{F}M$ itself. What is important to realize, though, is that $\mathscr{F}$ is a *functor*, i.e. the object $(\mathscr{F}M, M, \pi)$ *together with* a *particular* class of "naturally induced" morphisms.

Definition 1.6.5. Given a vector field $\xi \in \mathfrak{X}(M)$ and a natural bundle $\mathscr{F}$, we can define the ***natural lift*** $\mathscr{F}\xi \in \mathfrak{X}(\mathscr{F}M)$ of ξ as

$$\mathscr{F}\xi := \frac{\partial}{\partial t}\mathscr{F}(\varphi_t)\bigg|_{t=0}, \tag{1.6.3}$$

where $\{\varphi_t\}$ is the flow of ξ.

Roughly speaking, $\mathscr{F}\xi$ is obtained by differentiating the bundle transition functions with respect to the parameter time t. We shall explain this by means of two fundamental examples: the natural lifts onto the tangent and cotangent bundles.

The transition functions of the tangent [cotangent] bundle are essentially given by eqs. (1.6.1) [(1.6.2)]. Set then

$$x'^{\lambda} := \varphi^{\lambda}_t(x^{\mu}) = x^{\lambda} + t\xi^{\lambda}(x^{\mu}) + O^{\lambda}(t^2) \iff x^{\lambda} \equiv \varphi^{\lambda}_{-t}(x'^{\mu}) = x'^{\lambda} - t\xi^{\lambda}(x'^{\mu}) + O^{\lambda}(t^2)$$

and

$$y'^{\mu}(x^{\lambda}, y^{\nu}) = y^{\mu} + t\tilde{\xi}^{\mu}(x^{\lambda}, y^{\nu}) + O^{\mu}(t^2) \quad [\, y'_{\mu}(x'^{\lambda}, y_{\nu}) = y_{\mu} + t\tilde{\xi}_{\mu}(x'^{\lambda}, y_{\nu}) + O_{\mu}(t^2)\,].$$

On substituting these expressions into (1.6.1*b*) [(1.6.2*b*)], we get

$$\begin{aligned} y^{\mu} + t\tilde{\xi}^{\mu} + O^{\mu}(t^2) &= (\delta^{\mu}{}_{\nu} + t\partial_{\nu}\xi^{\mu} + \partial_{\nu}O^{\mu}(t^2))y^{\nu}, \\ [\, y_{\mu} + t\tilde{\xi}_{\mu} + O_{\mu}(t^2) &= (\delta^{\nu}{}_{\mu} - t\partial'_{\mu}\xi^{\nu} + \partial'_{\mu}O^{\nu}(t^2))y_{\nu}\,]. \end{aligned}$$

On differentiating the previous equation with respect to t at $t = 0$, one gets immediately

$$\tilde{\xi}^{\mu} = y^{\nu}\partial_{\nu}\xi^{\mu} \quad [\, \tilde{\xi}_{\mu} = -y_{\nu}\partial_{\mu}\xi^{\nu}\,], \tag{1.6.4}$$

or

$$T\xi = \xi^{\lambda}\frac{\partial}{\partial x^{\lambda}} + y^{\nu}\partial_{\nu}\xi^{\mu}\frac{\partial}{\partial y^{\mu}} \quad \left[T^*\xi = \xi^{\lambda}\frac{\partial}{\partial x^{\lambda}} - y_{\nu}\partial_{\mu}\xi^{\nu}\frac{\partial}{\partial y_{\mu}}\right],$$

which is the natural lift of ξ onto the tangent [cotangent] bundle.

Remark 1.6.6. This derivation is easily extended to any tensor (density) bundle.

Proposition 1.6.7. *Let ξ and η be two vector fields on M. Then,*

$$[\mathscr{F}\xi, \mathscr{F}\eta] = \mathscr{F}[\xi, \eta].$$

Proof (Kolář *et al.* 1993). First, note that $\mathscr{F}$ induces a smooth mapping between the appropriate spaces of local diffeomorphisms, which are infinite-dimensional manifolds.

Then, apply $\mathscr{F}$ to the curves (B.2.1) of Appendix B to get:

$$\begin{aligned}
\tfrac{\partial}{\partial t}(\varphi^{\mathscr{F}\eta}_{-t}\circ\varphi^{\mathscr{F}\xi}_{-t}\circ\varphi^{\mathscr{F}\eta}_{t}\circ\varphi^{\mathscr{F}\xi}_{t})\Big|_{t=0} &= 0,\\
[\mathscr{F}\xi,\mathscr{F}\eta] &= \tfrac{1}{2}\tfrac{\partial^2}{\partial t^2}(\varphi^{\mathscr{F}\eta}_{-t}\circ\varphi^{\mathscr{F}\xi}_{-t}\circ\varphi^{\mathscr{F}\eta}_{t}\circ\varphi^{\mathscr{F}\xi}_{t})\Big|_{t=0}\\
&= \tfrac{1}{2}\tfrac{\partial^2}{\partial t^2}\mathscr{F}(\varphi^{\eta}_{-t}\circ\varphi^{\xi}_{-t}\circ\varphi^{\eta}_{t}\circ\varphi^{\xi}_{t})\Big|_{t=0}\\
&= \tfrac{\partial}{\partial t}\mathscr{F}(\varphi^{[\xi,\eta]}_{t})\Big|_{t=0}\equiv\mathscr{F}[\xi,\eta],
\end{aligned}$$

which proves the proposition. □

1.7 Jets

Definition 1.7.1. Two curves $\gamma,\delta\colon\mathbb{R}\to M$ are said to have ***contact of order*** k ***at zero*** if, for every smooth function $f\colon M\to\mathbb{R}$, all derivatives up to order k of the difference $f\circ\gamma-f\circ\delta$ vanish at $0\in\mathbb{R}$.

In such a case, we write $\gamma\sim_k\delta$. Obviously, $\sim_k$ is an equivalence relation.

Definition 1.7.2. Two maps $\varphi,\varphi'\colon M\to N$ are then said to determine the same k***-jet*** at $x\in M$ if, for every curve $\gamma\colon\mathbb{R}\to M$ such that $\gamma(0)=x$, the curves $\varphi\circ\gamma$ and $\varphi'\circ\gamma$ have contact of order k at zero, and we shall write $j^k_x\varphi=j^k_x\varphi'$.

Now, let (B,M,π) be a fibred manifold.

Definition 1.7.3. The set J^kB of all k-jets of the local sections of (B,M,π) has a natural topology of a fibred manifold over M, denoted by (J^kB,M,π^k) and called the k***-th order jet prolongation*** of (B,M,π).

Remark 1.7.4. If (B,M,π) is a bundle, so is (J^kB,M,π^k).

The adapted fibred chart on J^kB induced by the chart $(V,x^\lambda,y^{\mathfrak{a}})$ on B will be denoted by $(J^kV,x^\lambda,y^{\mathfrak{a}}{}_{\boldsymbol{\mu}})$, where $\boldsymbol{\mu}$ is a multi-index of length $|\boldsymbol{\mu}|$ such that $0\leqslant|\boldsymbol{\mu}|\leqslant k$. Moreover, we shall set $\partial_{\mathfrak{a}}{}^{\boldsymbol{\mu}}:=\partial/\partial y^{\mathfrak{a}}{}_{\boldsymbol{\mu}}$ for $|\boldsymbol{\mu}|\geqslant 0$, and $\partial_{\boldsymbol{\mu}}:=\partial_{\mu_s}\circ\cdots\circ\partial_{\mu_1}$ for $\boldsymbol{\mu}=(\mu_s,\ldots,\mu_1)$.

If $\sigma\colon M\to B$ is a section of (B,M,π), its *k-th order jet prolongation* is the section $j^k\sigma$ of (J^kB,M,π^k) locally given by

$$y^{\mathfrak{a}}{}_{\boldsymbol{\mu}}\circ j^k\sigma:=\partial_{\boldsymbol{\mu}}\sigma^{\mathfrak{a}}\tag{1.7.1}$$

where $1\leqslant|\boldsymbol{\mu}|\leqslant k$ and $\sigma^{\mathfrak{a}}:=y^{\mathfrak{a}}\circ\sigma$. Vice versa, a section of (J^kB,M,π^k) which is the k-th order jet prolongation of some section $\sigma\colon M\to B$ is called a *holonomic section*. Furthermore, if $\Phi\colon B\to B'$ is a fibred morphism between two fibred manifolds (B,M,π) and (B',M',π') over a diffeomorphism $\varphi\colon M\to M'$, we define its *k-th order jet prolongation* $J^k\Phi\colon B\to B'$ by requiring

$$J^k\Phi\circ j^k\sigma=j^k(\Phi\circ\sigma\circ\varphi^{-1})\circ\varphi\tag{1.7.2}$$

for all sections $\sigma\colon M\to B$. It is easy to realize that, with (1.7.2), $J^k\colon\boldsymbol{FM}\to\boldsymbol{FM}$ becomes a functor in the sense of Definition A.2.1, and hence a natural bundle known as the *k-th order jet bundle*. We shall denote by π^k_h the canonical projections from J^kB to J^hB for $k > h$ and set $J^0B := B$. It is easy to see that $(J^kB, J^{k-1}B, \pi^k_{k-1})$ is a (natural) affine bundle modelled on the vector bundle $\bigvee^k T^*M \otimes_{J^{k-1}B} VB$, '$\bigvee^k$' denoting the k-th symmetric tensor product.

1.8 Horizontal and vertical differential

A form $\omega \in \Omega^p(J^kB)$, $k \geqslant 1$, is called a *contact p-form* if

$$(j^k\sigma)^*\omega = 0 \tag{1.8.1}$$

for any section σ of (B, M, π). Contact 1-forms are linear combinations of the *basis contact forms* $(\vartheta^{\mathfrak{a}}{}_{\boldsymbol{\mu}})$ of J^kB, defined as

$$\vartheta^{\mathfrak{a}}{}_{\boldsymbol{\mu}} := \mathrm{d}y^{\mathfrak{a}}{}_{\boldsymbol{\mu}} - y^{\mathfrak{a}}{}_{\boldsymbol{\mu}\nu}\,\mathrm{d}x^\nu,$$

$0 \leqslant |\boldsymbol{\mu}| \leqslant k-1$ [*cf.*, e.g., (1.8.3) and (1.8.21) below]. By a *horizontal p-form* on (B, M, π) we shall mean any form $\omega \in \Omega^p(B)$ such that

$$\omega(\underset{1}{\Upsilon}, \ldots, \underset{p}{\Upsilon}) = 0,$$

$\{\underset{i}{\Upsilon}\}$ being vertical vector fields (*cf.* §1.2). Locally, a horizontal p-form ω reads

$$\omega \equiv \frac{1}{p!}\omega_{\alpha_1\ldots\alpha_p}\,\mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_p}. \tag{1.8.2}$$

Horizontal p-forms span a subspace $\Omega^p_0(B)$ of $\Omega^p(B)$.

If $\omega \in \Omega^1(B)$, on J^1B we can decompose it as follows:

$$\begin{aligned}(\pi^1_0)^*\omega &\equiv \omega_\mu\,\mathrm{d}x^\mu + \omega_{\mathfrak{a}}\,\mathrm{d}y^{\mathfrak{a}}\\ &= \omega_\mu\,\mathrm{d}x^\mu + \omega_{\mathfrak{a}}(\mathrm{d}y^{\mathfrak{a}} - y^{\mathfrak{a}}{}_\mu\,\mathrm{d}x^\mu + y^{\mathfrak{a}}{}_\mu\,\mathrm{d}x^\mu)\\ &= (\omega_\mu + \omega_{\mathfrak{a}}\,y^{\mathfrak{a}}{}_\mu)\,\mathrm{d}x^\mu + \omega_{\mathfrak{a}}\vartheta^{\mathfrak{a}},\end{aligned} \tag{1.8.3}$$

namely we can express $(\pi^1_0)^*\omega$ as the sum of a contact 1-form $\omega_{\mathfrak{a}}\vartheta^{\mathfrak{a}}$ and a horizontal 1-form on J^1B,

$$h(\omega) := (\omega_\mu + \omega_{\mathfrak{a}}\,y^{\mathfrak{a}}{}_\mu)\,\mathrm{d}x^\mu, \tag{1.8.4}$$

called the *horizontal part* of ω. Such a definition can be extended to 1-forms on any jet prolongation of B, i.e. to forms $\omega \in \Omega^1(J^kB)$. If

$$\omega \equiv \omega_\nu\,\mathrm{d}x^\nu + \omega_{\mathfrak{a}}{}^{\boldsymbol{\mu}}\,\mathrm{d}y^{\mathfrak{a}}{}_{\boldsymbol{\mu}}$$

is a 1-form on J^kB, one defines the operator $h\colon \Omega^1(J^kB) \to \Omega^1_0(J^{k+1}B)$ as

$$h(\omega) := (\omega_\nu + \omega_{\mathfrak{a}}{}^{\boldsymbol{\mu}}\,y^{\mathfrak{a}}{}_{\boldsymbol{\mu}\nu})\,\mathrm{d}x^\nu. \tag{1.8.5}$$

It is possible to extend the previous definition also to 0-forms, i.e. to smooth functions from J^kB to $\mathbb{R}$, by setting

$$h(f) := f, \tag{1.8.6}$$

for any $f \in \Omega^0(J^kB) \equiv C^\infty(J^kB;\mathbb{R})$. Finally, we can extend the definition of h to *any* p-form by using the fact that there exists a unique linear operator

$$h\colon \Omega^p(J^kB) \to \Omega^p_0(J^{k+1}B)$$

that coincides with the previous definitions for $p = 0$ and $p = 1$, and satisfies the property

$$h(\omega \wedge \chi) = h(\omega) \wedge h(\chi) \tag{1.8.7}$$

for any two forms ω e χ. Furthermore, the *contact part* $k(\omega)$ of a form $\omega \in \Omega^p(J^kB)$, defined as

$$k(\omega) := (\pi^{k+1}_k)^*\omega - h(\omega),$$

is *always* a contact form. Therefore,

$$(j^{k+1}\sigma)^*\omega = (j^{k+1}\sigma)^* h(\omega) \tag{1.8.8}$$

by (1.8.1), where, on the l.h.s., we simply wrote ω for $(\pi^{k+1}_k)^*\omega$, it being hereafter understood that a form $\omega \in \Omega^p(J^kB)$ need first be pulled back onto $J^{k+1}B$ to be decomposed into its horizontal and contact parts [*cf.* (1.8.3)].

The ***horizontal differential*** $\mathrm{d_H}\colon \Omega^p_0(J^kB) \to \Omega^{p+1}_0(J^{k+1}B)$ is defined as

$$\mathrm{d_H}\omega := h(\mathrm{d}\omega). \tag{1.8.9}$$

for any $\omega \in \Omega^p_0(J^kB)$ and is such that

$$\mathrm{d_H}(\omega \wedge \chi) = \mathrm{d_H}\omega \wedge \chi + (-1)^p \omega \wedge \mathrm{d_H}\chi. \tag{1.8.10}$$

If we define the ***formal*** (or ***total***) ***derivative*** $\mathrm{d}_\nu\colon \Omega^0(J^kB) \to \Omega^0(J^{k+1}B)$ as

$$\mathrm{d}_\nu f := \partial_\nu f + y^{\mathfrak{a}}{}_{\mu\nu}\partial_{\mathfrak{a}}{}^{\mu} f, \tag{1.8.11}$$

for any function $f \in \Omega^0(J^kB)$, then we have

$$\mathrm{d_H} f = \mathrm{d}_\mu f\, \mathrm{d}x^\mu \tag{1.8.12}$$

and

$$\mathrm{d}_\mu \mathrm{d}_\nu f = \mathrm{d}_\nu \mathrm{d}_\mu f. \tag{1.8.13}$$

More generally, if $\omega \in \Omega^p_0(J^kB)$, one has [*cf.* (1.8.2)]

$$\mathrm{d_H}\omega = \frac{1}{p!}\mathrm{d}_\mu \omega_{\alpha_1\ldots\alpha_p}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^{\alpha_1} \wedge \cdots \wedge \mathrm{d}x^{\alpha_p}. \tag{1.8.14}$$

Thus, property (1.8.13) entails

$$\mathrm{d_H}\mathrm{d_H}\omega = 0 \tag{1.8.15}$$

if ω is a horizontal form (or a function).

Now, it turns out that any p-form on J^kB can be expressed as a linear combination of exterior products of horizontal forms and basis contact forms. Therefore, if we define

$$\mathrm{d_H}\vartheta^{\mathfrak{a}}{}_{\boldsymbol{\mu}} := \mathrm{d}\vartheta^{\mathfrak{a}}{}_{\boldsymbol{\mu}}$$

and require that $\mathrm{d_H}$ be linear and satisfy property (1.8.10) for any pair of forms ω and χ, the definition of horizontal differential can be uniquely extended to any p-form. Furthermore, because of the way in which we defined the horizontal differential on contact forms and of its properties on horizontal forms, identity (1.8.15) holds also for any p-form $\omega \in \Omega^p(J^kB)$.

Finally, note that horizontal differential and formal derivative are defined in such a way that, if σ is a section of a fibred manifold (B, M, π), then

$$(j^{k+1}\sigma)^*\mathrm{d_H}\omega = (j^k\sigma)^*\mathrm{d}\omega \equiv \mathrm{d}[(j^k\sigma)^*\omega]$$

for any $\omega \in \Omega^p(J^kB)$, and

$$(\mathrm{d}_\mu f) \circ j^{k+1}\sigma = \partial_\mu(f \circ j^k\sigma) \tag{1.8.16}$$

for any $f \in \Omega^0(J^kB)$. Moreover,

$$h \circ \mathrm{d} \equiv \mathrm{d_H} \circ h. \tag{1.8.17}$$

We can now define the ***vertical differential*** $\mathrm{d_V}\colon \Omega^p(J^kB) \to \Omega^{p+1}(J^{k+1}B)$ as the difference between the standard and the horizontal differential, i.e.

$$\mathrm{d_V}\omega := \mathrm{d}\omega - \mathrm{d_H}\omega \tag{1.8.18}$$

for any $\omega \in \Omega^p(J^kB)$. It is immediate to see that the vertical differentials of the base coordinates (x^λ) vanish and the vertical differentials of the coordinates $(y^{\mathfrak{a}}{}_{\boldsymbol{\mu}})$ are nothing but the base contact forms $(\vartheta^{\mathfrak{a}}{}_{\boldsymbol{\mu}})$. Furthermore, the vertical differential of any p-form is always a contact form, and the horizontal part of a vertical differential always vanishes. Finally, from the properties of the standard and the horizontal differential it follows that the vertical differential is linear and moreover one has

$$\mathrm{d_V}(\omega \wedge \chi) = \mathrm{d_V}\omega \wedge \chi + (-1)^p \omega \wedge \mathrm{d_V}\chi$$

for any $\omega \in \Omega^p(J^kB)$ and

$$\mathrm{d_V}\mathrm{d_V}\omega = 0.$$

1.8.1 Examples

For the reader's convenience, we shall now give some examples of formal derivatives, horizontal and vertical differentials which often recur in this thesis. First of all, consider the formal derivative of $y^{\mathfrak{a}}$. On applying (1.8.11), we obtain immediately

$$\mathrm{d}_\mu y^{\mathfrak{a}} = y^{\mathfrak{b}}{}_\mu \frac{\partial y^{\mathfrak{a}}}{\partial y^{\mathfrak{b}}} = y^{\mathfrak{a}}{}_\mu. \tag{1.8.19}$$

On using (1.8.12), we also have

$$\mathrm{d_H} y^{\mathfrak{a}} = \mathrm{d}_\mu y^{\mathfrak{a}} \,\mathrm{d}x^\mu = y^{\mathfrak{a}}{}_\mu \,\mathrm{d}x^\mu. \tag{1.8.20}$$

Hence, on applying (1.8.18), we get effortlessly

$$\mathrm{d_V} y^{\mathfrak{a}} = \mathrm{d}y^{\mathfrak{a}} - \mathrm{d_H} y^{\mathfrak{a}} = \mathrm{d}y^{\mathfrak{a}} - y^{\mathfrak{a}}{}_\mu \,\mathrm{d}x^\mu, \tag{1.8.21}$$

which is nothing but the basis contact form $\vartheta^{\mathfrak{a}}$, as anticipated. □

Now, we have already seen that for a form $\omega \in \Omega^1(B)$ decomposition (1.8.3) holds. The contact part of ω is given by

$$k(\omega) = \omega_{\mathfrak{a}} \vartheta^{\mathfrak{a}} \equiv \omega_{\mathfrak{a}} (\mathrm{d}y^{\mathfrak{a}} - y^{\mathfrak{a}}{}_\mu \,\mathrm{d}x^\mu).$$

Let us verify important property (1.8.1), then. By virtue of (1.7.1) we have

$$(j^1\sigma)^* k(\omega) = \omega_{\mathfrak{a}}(\sigma)(\partial_\mu \sigma^{\mathfrak{a}} \,\mathrm{d}x^\mu - \partial_\mu \sigma^{\mathfrak{a}} \,\mathrm{d}x^\mu) = 0,$$

as claimed. So, property (1.8.8) holds, but we can also check it directly. Indeed, the horizontal part of ω is given by (1.8.4). Thus,

$$(j^1\sigma)^* h(\omega) = [\omega_\mu(\sigma) + \omega_{\mathfrak{a}}(\sigma)\partial_\mu \sigma^{\mathfrak{a}}] \,\mathrm{d}x^\mu$$

by (1.7.1). On the other hand,

$$(j^1\sigma)^* \omega \equiv (j^1\sigma)^* (\omega_\mu \,\mathrm{d}x^\mu + \omega_{\mathfrak{a}} \,\mathrm{d}y^{\mathfrak{a}}) = [\omega_\mu(\sigma) + \omega_{\mathfrak{a}}(\sigma)\partial_\mu \sigma^{\mathfrak{a}}] \,\mathrm{d}x^\mu$$

directly from the fact that $y^{\mathfrak{a}} \circ \sigma = \sigma^{\mathfrak{a}}$. □

Finally, let us check property (1.8.17). Let $\omega \in \Omega^1(B)$ be as in (1.8.3). Now, the horizontal part of ω is given by (1.8.4). Its horizontal differential is

$$\begin{aligned} \mathrm{d_H} h(\omega) &= (\partial_\nu \omega_\mu + y^{\mathfrak{a}}{}_\mu \,\mathrm{d}_\nu \omega_{\mathfrak{a}} + \omega_{\mathfrak{a}}\, y^{\mathfrak{a}}{}_{\mu\nu}) \,\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu + \\ &= (\partial_\nu \omega_\mu + y^{\mathfrak{a}}{}_\mu \,\mathrm{d}_\nu \omega_{\mathfrak{a}}) \,\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \end{aligned}$$

by virtue of (1.8.5) and the symmetry of $y^{\mathfrak{a}}{}_{\mu\nu}$. On the other hand,

$$\begin{aligned} h(\mathrm{d}\omega) &= h(\partial_\nu \omega_\mu \,\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu + \partial_\nu \omega_{\mathfrak{a}} \,\mathrm{d}x^\nu \wedge \mathrm{d}y^{\mathfrak{a}} + \partial_{\mathfrak{b}} \omega_{\mathfrak{a}} \,\mathrm{d}y^{\mathfrak{b}} \wedge \mathrm{d}y^{\mathfrak{a}}) \\ &= (\partial_\nu \omega_\mu + y^{\mathfrak{a}}{}_\mu \partial_\nu \omega_{\mathfrak{a}} + y^{\mathfrak{a}}{}_\mu y^{\mathfrak{b}}{}_\nu \partial_{\mathfrak{b}} \omega_{\mathfrak{a}}) \,\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu \\ &= (\partial_\nu \omega_\mu + y^{\mathfrak{a}}{}_\mu \,\mathrm{d}_\nu \omega_{\mathfrak{a}}) \,\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu, \end{aligned}$$

where we used properties (1.8.7) and (1.8.6), eq. (1.8.20) and definition (1.8.11). □

1.9 First order jet bundle

In section §1.7 we introduced the concept of the k-th order jet bundle over a fibre manifold (B, M, π). In the sequel, we shall be mainly concerned with first order jets. Fibred coordinates on J^1B will be denoted by $(x^\lambda, y^{\mathfrak{a}}, y^{\mathfrak{a}}{}_\mu)$ and the transition functions are clearly given by the transformation rule for first order derivatives, namely

$$\begin{aligned} x'^\lambda &= x'^\lambda(x^\mu) \\ y'^{\mathfrak{a}} &= y'^{\mathfrak{a}}(x^\mu, y^{\mathfrak{b}}) \\ y'^{\mathfrak{a}}{}_\mu &= \frac{\partial x^\nu}{\partial x'^\mu}\left(\frac{\partial y'^{\mathfrak{a}}}{\partial x^\nu} + \frac{\partial y'^{\mathfrak{a}}}{\partial y^{\mathfrak{b}}} y^{\mathfrak{b}}{}_\nu\right) \\ &\equiv \frac{\partial x^\nu}{\partial x'^\mu}\, \mathrm{d}_\nu y'^{\mathfrak{a}}, \end{aligned} \tag{1.9.1}$$

where we used (1.8.11). Therefore, if we have a vector field Ξ on B, i.e. locally

$$\Xi(x,y) = \Xi^\mu(x^\lambda, y^{\mathfrak{b}})\frac{\partial}{\partial x^\mu} + \Xi^{\mathfrak{a}}(x^\lambda, y^{\mathfrak{b}})\frac{\partial}{\partial y^{\mathfrak{a}}},$$

we can compute its natural lift $J^1\Xi$ according to Definition 1.6.5. Set then

$$x'^\lambda = x^\lambda + t\Xi^\lambda + O^\lambda(t^2), \tag{1.9.2a}$$
$$y'^{\mathfrak{a}} = y^{\mathfrak{a}} + t\Xi^{\mathfrak{a}} + O^{\mathfrak{a}}(t^2), \tag{1.9.2b}$$
$$y'^{\mathfrak{a}}{}_\mu = y^{\mathfrak{a}}{}_\mu + t\Xi^{\mathfrak{a}}{}_\mu + O^{\mathfrak{a}}{}_\mu(t^2). \tag{1.9.2c}$$

Now, substitute eqs. (1.9.2) into (1.9.1) to get

$$y^{\mathfrak{a}}{}_\mu + t\Xi^{\mathfrak{a}}{}_\mu + O^{\mathfrak{a}}{}_\mu(t^2) = (\delta^\nu{}_\mu - t\,\partial'_\mu\Xi^\nu + \partial'_\mu O^\nu(t^2))(y^{\mathfrak{a}}{}_\nu + t\,\mathrm{d}_\nu\Xi^{\mathfrak{a}} + \mathrm{d}_\nu O^{\mathfrak{a}}(t^2)),$$

where we used (1.8.19). Differentiating the last equation with respect to t at $t = 0$ gives

$$\Xi^{\mathfrak{a}}{}_\mu = \mathrm{d}_\mu\Xi^{\mathfrak{a}} - y^{\mathfrak{a}}{}_\nu \mathrm{d}_\mu\Xi^\nu$$

or

$$J^1\Xi = \Xi^\mu\frac{\partial}{\partial x^\mu} + \Xi^{\mathfrak{a}}\frac{\partial}{\partial y^{\mathfrak{a}}} + (\mathrm{d}_\mu\Xi^{\mathfrak{a}} - y^{\mathfrak{a}}{}_\nu\mathrm{d}_\mu\Xi^\nu)\frac{\partial}{\partial y^{\mathfrak{a}}{}_\mu}. \tag{1.9.3}$$

1.10 Gauge-natural bundles

The category of natural bundles is not large enough to encompass all fibre bundles appearing in classical field theory. In particular, principal bundles, which constitute the geometrical arena of gauge theory, are not, in general, natural bundles. To this end, a suitable generalization of the notion of a natural bundle must be given: this was accomplished by Eck (1981), who introduced the concept of a "gauge-natural bundle".

We shall now give both the axiomatic and the constructive definition of such a functor. This section closely follows Kolář *et al.* (1993), notably §15 and Chapter XII.

Definition 1.10.1. Let G be a Lie group, and $\boldsymbol{PB}_m(G)$ the category of principal G-bundles over m-dimensional manifolds and principal morphisms defined in §1.3. A ***gauge-natural bundle*** is a functor $\mathscr{F}\colon \boldsymbol{PB}_m(G) \to \boldsymbol{FM}$ such that:

(*i*) every principal bundle $(P, M, \pi; G) \in \mathrm{Ob}(\boldsymbol{PB}_m(G))$ is transformed into a fibred manifold $(\mathscr{F}P, M, \tilde{\pi}) \in \mathrm{Ob}(\boldsymbol{FM})$;

(*ii*) every principal morphism $\Phi\colon P \to P'$ between two principal bundles $P(M,G)$, $P'(M',G) \in \mathrm{Ob}(\boldsymbol{PB}_m(G))$ over a local diffeomorphism $\varphi\colon M \to M'$ is transformed into a fibred morphism $\mathscr{F}\Phi\colon \mathscr{F}P \to \mathscr{F}P'$ over φ;

(*iii*) for every principal bundle $(P, M, \pi; G) \in \mathrm{Ob}(\boldsymbol{PB}_m(G))$ and every open subset $U \subseteq M$ the inclusion $\iota\colon \pi^{-1}(U) \to P$ is transformed into the inclusion $\mathscr{F}\iota\colon \tilde{\pi}^{-1}(U) \to \mathscr{F}P$.

A section of $(\mathscr{F}P, M, \tilde{\pi})$ is called a *gauge-natural object.*

The choice $G = \{e\}$, where e denotes the unit element of G, reproduces, of course, the natural bundles on $\boldsymbol{Mf}_m$ as defined in §1.6.

We shall now give a different characterization of (gauge-) natural bundles as fibre bundles associated with a particular class of principal bundles (together with a special class of morphisms). To this end, we need first to introduce a few preliminary concepts.

Definition 1.10.2. The set

$$\{\, j^k_0\alpha \mid \alpha\colon \mathbb{R}^m \to \mathbb{R}^m,\ \alpha(0) = 0, \text{ locally invertible}\,\}$$

equipped with the jet composition $j^k_0\alpha \circ j^k_0\alpha' := j^k_0(\alpha \circ \alpha')$ is a Lie group called the ***k-th differential group*** and denoted by G^k_m.

For $k = 1$ we have, of course, the identification $G^1_m \cong \mathrm{GL}(m, \mathbb{R})$.

Definition 1.10.3. Let M be an m-dimensional manifold. The principal bundle over M with group G^k_m is called the ***bundle of k-frames (over M)*** and will be denoted by L^kM.

For $k = 1$ we have, of course, the identification $L^1M \cong LM$, where LM is the bundle of linear frames over M (*cf.* §1.3.1).

Definition 1.10.4. Let G be a Lie group. Then, by the ***space of (m, h)-velocities*** of G we shall mean the set

$$T^h_m G := \{\, j^h_0 a \mid a\colon \mathbb{R}^m \to G\,\}.$$

Thus, $T^h_m G$ denotes the set of h-jets with "source" at the origin $0 \in \mathbb{R}^m$ and "target" in G, and can be given the structure of a (Lie) group. Indeed, let $S, T \in T^h_m G$ be any elements. We define a (smooth) multiplication in $T^h_m G$ as

$$\begin{cases} T^h_m\mu\colon T^h_m G \times T^h_m G \to T^h_m G \\ T^h_m\mu\colon (S = j^h_0 a, T = j^h_0 b) \mapsto S \cdot T := j^h_0(ab) \end{cases},$$

where $(ab)(x) := a(x)b(x) \equiv \mu(a(x), b(x))$ is the group multiplication in G. The mapping $(S, T) \mapsto S \cdot T$ is associative; moreover, the element $j^h_0 e$, e denoting both the unit element in G and the constant mapping from $\mathbb{R}^m$ onto e, is the unit element of $T^h_m G$, and $j^h_0 a^{-1}$, where $a^{-1}(x) := a(x)^{-1}$ (the inversion being taken in the group G), is the inverse element of $j^h_0 a$.

Definition 1.10.5. Consider a principal bundle $P(M, G)$. Let k and h be two natural numbers such that $k \geqslant h$. Then, by the (k, h)***-principal prolongation*** of P we shall mean the bundle

$$W^{k,h}P := L^kM \times_M J^hP. \tag{1.10.1}$$

A point of $W^{k,h}P$ is of the form $(j_0^k\epsilon, j_x^h\sigma)$, where $\epsilon\colon \mathbb{R}^m \to M$ is locally invertible and such that $\epsilon(0) = x$, and $\sigma\colon M \to P$ is a local section around the point $x \in M$.

Unlike J^hP, $W^{k,h}P$ is a principal bundle over M, and its structure group is

$$W_m^{k,h}G := G_m^k \rtimes T_m^hG.$$

$W_m^{k,h}G$ is called the $(m; k, h)$*-principal prolongation* of G. The group multiplication on $W_m^{k,h}G$ is defined by the following rule:

$$(j_0^k\alpha, j_0^h a) \odot (j_0^k\beta, j_0^h b) := \Big(j_0^k(\alpha \circ \beta), j_0^h\big((a \circ \beta)b\big)\Big).$$

The right action of $W_m^{k,h}G$ on $W^{k,h}P$ is then defined by

$$(j_0^k\epsilon, j_x^h\sigma) \odot (j_0^k\alpha, j_0^h a) := \Big(j_0^k(\epsilon \circ \alpha), j_x^h\big(\sigma \cdot (a \circ \alpha^{-1} \circ \epsilon^{-1})\big)\Big), \tag{1.10.2}$$

'$\cdot$' denoting the canonical right action of G on P.

In the case $h = 0$, we have a direct product of Lie groups $W_m^{k,0}G := G_m^k \times G$ and the usual fibred product $W^{k,0}P \equiv L^kM \times_M P$ of principal bundles.

Definition 1.10.6. Let $\Phi\colon P \to P$ be an automorphism over a diffeomorphism $\varphi\colon M \to M$ (*cf.* §1.3). We define an ***automorphism*** of $W^{k,h}P$ associated with Φ as

$$\begin{cases} W^{k,h}\Phi\colon W^{k,h}P \to W^{k,h}P \\ W^{k,h}\Phi\colon (j_0^k\epsilon, j_x^h\sigma) \mapsto \big(j_0^k(\varphi \circ \epsilon), j_x^h(\Phi \circ \sigma \circ \varphi^{-1})\big) \end{cases}. \tag{1.10.3}$$

Proposition 1.10.7. *The bundle morphism $W^{k,h}\Phi$ preserves the right action, thereby being a principal automorphism.*

Proof. We have:

$$\begin{aligned} W^{k,h}\Phi(j_0^k\epsilon, j_x^h\sigma) \odot (j_0^k\alpha, j_0^h a) &= \big(j_0^k(\varphi \circ \epsilon), j_x^h(\Phi \circ \sigma \circ \varphi^{-1})\big) \odot (j_0^k\alpha, j_0^h a) \\ &= \Big(j_0^k\big((\varphi \circ \epsilon) \circ \alpha\big), j_x^h\big(\Phi \circ \sigma \circ \varphi^{-1} \cdot (a \circ \alpha^{-1} \circ (\varphi \circ \epsilon)^{-1})\big)\Big) \\ &= \Big(j_0^k(\varphi \circ \epsilon \circ \alpha), j_x^h\big(\Phi \circ \sigma \circ \varphi^{-1} \cdot (a \circ \alpha^{-1} \circ \epsilon^{-1} \circ \varphi^{-1})\big)\Big) \\ &= \Big(j_0^k\big(\varphi \circ (\epsilon \circ \alpha)\big), j_x^h\big(\Phi \circ \sigma \cdot (a \circ \alpha^{-1} \circ \epsilon^{-1}) \circ \varphi^{-1}\big)\Big) \\ &= W^{k,h}\Phi\Big((j_0^k\epsilon, j_x^h\sigma) \odot (j_0^k\alpha, j_0^h a)\Big), \end{aligned}$$

where in the first equality we used (1.10.3), in the second one (1.10.2), and in the last one both. Therefore, $W^{k,h}\Phi$ preserves the right action. By Definition 1.3.2 this means that $W^{k,h}\Phi$ is a principal automorphism of $W^{k,h}P$. □

By virtue of (1.10.1) and (1.10.3) $W^{k,h}$ turns out to be a functor from the category of principal G-bundles over m-dimensional manifolds and local isomorphisms to the category of principal $W^{k,h}_m G$-bundles. Now, let $P_\lambda := W^{k,h}P \times_\lambda F$ be a fibre bundle associated with $P(M,G)$ via an action λ of $W^{k,h}_m G$ on a manifold F. There exists a canonical representation of the automorphisms of P induced by (1.10.3). Indeed, if $\Phi\colon P \to P$ is an automorphism over a diffeomorphism $\varphi\colon M \to M$, then we can define the corresponding *induced automorphism* Φ_λ as

$$\begin{cases} \Phi_\lambda\colon P_\lambda \to P_\lambda \\ \Phi_\lambda\colon [u,f]_\lambda \mapsto [W^{k,h}\Phi(u), f]_\lambda \end{cases}, \tag{1.10.4}$$

which is well-defined since it is independent of the representative $(u,f) \in P \times F$. Indeed, if $(u',f') \in [u,f]_\lambda$, then by (1.4.1) $u' = u \cdot a$ and $f' = a^{-1} \cdot f$ for some $a \in G$. Therefore,

$$\begin{aligned} [W^{k,h}\Phi(u'), f']_\lambda &= [W^{k,h}\Phi(u \cdot a), a^{-1} \cdot f]_\lambda \\ &= [W^{k,h}\Phi(u) \cdot a, a^{-1} \cdot f]_\lambda \\ &= [W^{k,h}\Phi(u), f]_\lambda, \end{aligned}$$

where the second equality follows from Proposition 1.10.7, and the third one from using (1.4.1) once again. □

This construction yields a functor $\cdot_\lambda$ from the category of principal G-bundles to the category of fibred manifolds and fibre-respecting mappings.

Definition 1.10.8. A ***gauge-natural bundle of order*** (k,h) over M associated with $P(M,G)$ is any such functor.

It can be shown that this constructive definition of gauge-natural bundles is indeed equivalent to the axiomatic definition given above (*cf.* Kolář *et al.* 1993, §51): notably, $\cdot_\lambda \cong \mathscr{F}$, where the symbol '$\cong$' is to be understood in the sense of a natural isomorphism as per Definition A.2.4. This important result is usually expressed by saying that *gauge-natural bundles have finite order*.

Now, we saw earlier that natural bundles are gauge-natural bundles with $G = \{e\}$. Indeed, we have the following

Definition 1.10.9. Let $\varphi\colon M \to M$ be a diffeomorphism. We define an automorphism of L^kM associated with φ, called its ***natural lift onto*** L^kM, by

$$\begin{cases} L^k\varphi\colon L^kM \to L^kM \\ L^k\varphi\colon j^k_0\epsilon \mapsto j^k_0(\varphi \circ \epsilon) \end{cases}. \tag{1.10.5}$$

Then, L^k turns out to be a functor from the category of m-dimensional manifolds and local diffeomorphisms to the category of principal G^k_m-bundles. Now, given any fibre bundle associated with L^kM and any diffeomorphism on M, we can define a corresponding induced automorphism along the lines of (1.10.4). This construction yields a functor from the category of m-dimensional manifolds to the category of fibred manifolds.

Definition 1.10.10. A ***natural bundle of order*** k over M is any such functor.

Remark 1.10.11. Unless explicitly stated otherwise, in the sequel we shall always assume that L^kM is equipped with the principal bundle structure *naturally* induced by the differentiable structure of the base manifold M, i.e. that L^kM itself is a natural bundle over M. This is, of course, possible because we can always identify a principal bundle $P(M, G)$ with its associated bundle $P_\lambda := P \times_\lambda G$, where λ is the left action of G on itself. Hence, we can regard L^kM as a principal bundle associated with itself, whose *only* automorphisms are of the type (1.10.5).

Definition 1.10.12. Let P_λ be a gauge-natural bundle associated with a principal bundle $P(M, G)$ and let Ξ be a G-invariant vector field on P. By the ***gauge-natural lift*** of Ξ onto P_λ we shall mean the vector field $\Xi_\lambda \in \mathfrak{X}(P_\lambda)$ defined as

$$\Xi_\lambda := \frac{\partial}{\partial t}(\Phi_t)_\lambda \bigg|_{t=0}, \tag{1.10.6}$$

$\{\Phi_t\}$ denoting the flow of Ξ.

Remark 1.10.13. Of course, if P_λ is a (purely) natural bundle, the gauge-natural lift of any vector field in $\mathfrak{X}(M)$ reduces to its natural lift as defined in §1.6.

Proposition 1.10.14. *Let Ξ and H be two G-invariant vector fields on a principal bundle $P(M, G)$. Then,*

$$[\Xi_\lambda, \mathrm{H}_\lambda] = [\Xi, \mathrm{H}]_\lambda.$$

Proof. It is enough to realize that the argument we used in proving Proposition 1.6.7 goes through unmodified if we replace M with P, and ξ and η with Ξ and H, respectively. □

1.10.1 Examples

We shall now give some important examples of (gauge-) natural bundles.

Example 1.10.15 (Bundle of tensor density fields). A first fundamental example of a natural bundle is given, of course, by the bundle ${}^wT^r_sM$ of tensor density fields of weight w over an m-dimensional manifold M. Indeed, ${}^wT^r_sM$ is a vector bundle associated with L^1M via the following left action of $G^1_m \cong W^{1,0}_m\{e\} \cong \mathrm{GL}(m, \mathbb{R})$ on the vector space $T^r_s(\mathbb{R}^m)$ [*cf.* (1.4.2)]:

$$\begin{cases} \lambda\colon G^1_m \times T^r_s(\mathbb{R}^m) \to T^r_s(\mathbb{R}^m) \\ \lambda\colon (\alpha^j{}_k, t^{p_1\dots p_r}_{q_1\dots q_s}) \mapsto \alpha^{p_1}{}_{k_1} \cdots \alpha^{p_r}{}_{k_r} t^{k_1\dots k_r}_{l_1\dots l_s} \tilde{\alpha}^{l_1}{}_{q_1} \cdots \tilde{\alpha}^{l_s}{}_{q_s} (\det \alpha)^{-w} \end{cases}.$$

For $w = 0$ we recover the bundle of tensor fields over M. This is a definition of ${}^wT^r_sM$ which is appropriate for physical applications, where one usually considers *only* those (active) transformations of tensor fields that are *naturally* induced by some transformations on the base manifold (*cf.* §1.6). Somewhat more unconventionally, though, we can regard ${}^wT^r_sM$ as a *gauge*-natural vector bundle associated with $W^{0,0}(LM)$. Also, in §1.4 we saw that ${}^wT^r_sM$ could be equally well regarded as a (gauge-natural vector) bundle associated with $\mathrm{SO}(M, g)$. Of course, the three bundles under consideration are the same *as objects*, but their *morphisms* are different. In Chapter 2 we shall see that this implies, in particular, that we have (at least) three *different* notions of a Lie derivative for a tensor (density) field.

Example 1.10.16 (Bundle of principal connections). Let $P(M,G)$ be a principal bundle, and $(\mathrm{Ad}_a)^{\mathcal{A}}{}_{\mathcal{B}}$ the coordinate expression of the adjoint representation of G. Set $\mathcal{A} := (\mathbb{R}^m)^* \otimes \mathfrak{g}$, where $\mathfrak{g}$ denotes the Lie algebra of G, and consider the action

$$\begin{cases} \ell\colon W^{1,1}_m G \times \mathcal{A} \to \mathcal{A} \\ \ell\colon \big((\alpha^j{}_k, a^{\mathcal{B}}, a^{\mathcal{C}}{}_l), w^{\mathcal{A}}{}_i\big) \mapsto (\mathrm{Ad}_a)^{\mathcal{A}}{}_{\mathcal{B}}(w^{\mathcal{B}}{}_j - a^{\mathcal{B}}{}_j)\tilde{\alpha}^j{}_i \end{cases}, \tag{1.10.7}$$

where $(a^{\mathcal{A}}, a^{\mathcal{B}}{}_i)$ denote natural coordinates on $T^1_m G$: a generic element $j^1_0 f \in T^1_m G$ is represented by $a = f(0) \in G$, i.e. $a^{\mathcal{A}} = f^{\mathcal{A}}(0)$, and $a^{\mathcal{B}}{}_i = (\partial_i(a^{-1}f(x))|_{x=0})^{\mathcal{B}}$. Comparing with (1.5.14) and noting that, there, we did not take into account transformations of the 1-form index μ, it is immediate to realize that the sections of $W^{1,1}P \times_\ell \mathcal{A}$ are in 1-1 correspondence with the principal connections on P. A section of $W^{1,1}P \times_\ell \mathcal{A}$ will be called a ***G*-*connection***. Clearly, $W^{1,1}P \times_\ell \mathcal{A}$ is a gauge-natural *affine* bundle of order $(1,1)$.

Example 1.10.17 (Bundle of linear connections). Of course, a $\mathrm{GL}(m,\mathbb{R})$-connection in the sense of the previous example is nothing but a linear connection on M. It is interesting to note, though, that in this case, i.e. when $P = LM$, a principal connection can be seen as a section of a (purely) *natural* bundle. Indeed, consider the following action of $G^2_m \cong W^{2,0}_m\{e\}$ on $T^1{}_2(\mathbb{R}^m) \cong (\mathbb{R}^m)^* \otimes \mathfrak{gl}(m,\mathbb{R})$

$$\begin{cases} \tilde{\ell}\colon G^2_m \times T^1{}_2(\mathbb{R}^m) \to T^1{}_2(\mathbb{R}^m) \\ \tilde{\ell}\colon \big((\alpha^l{}_m, \alpha^n{}_{pq}), w^i{}_{jk}\big) \mapsto \alpha^i{}_l w^l{}_{mn}\tilde{\alpha}^m{}_j\tilde{\alpha}^n{}_k + \alpha^i{}_l\tilde{\alpha}^l{}_{jk} \end{cases}, \tag{1.10.8}$$

which is clearly equivalent to (1.10.7) when $G = \mathrm{GL}(m,\mathbb{R})$ [*cf.* (1.5.15) and (1.5.16)]. Thus, $L^2M \times_{\tilde{\ell}} T^1{}_2(\mathbb{R}^m)$ can, and shall, be regarded as a natural affine bundle of order 2. A section of $L^2M \times_{\tilde{\ell}} T^1{}_2(\mathbb{R}^m)$ will be called a ***natural linear connection*** on M. Again, as in Example 1.10.15, the bundle of natural linear connections and the bundle of $\mathrm{GL}(m,\mathbb{R})$-connections are the same *as objects*, but their *morphisms* are different.

Example 1.10.18 (Bundle of *G*-invariant vector fields). Let $\mathcal{V} := \mathbb{R}^m \oplus \mathfrak{g}$, and consider the following action:

$$\begin{cases} \lambda\colon W^{1,1}_m G \times \mathcal{V} \to \mathcal{V} \\ \lambda\colon \big((\alpha^j{}_k, a^{\mathcal{B}}, a^{\mathcal{C}}{}_l), (v^i, w^{\mathcal{A}})\big) \mapsto \big(\alpha^i{}_j v^j, (\mathrm{Ad}_a)^{\mathcal{A}}{}_{\mathcal{B}}(w^{\mathcal{B}} + a^{\mathcal{B}}{}_j v^j)\big) \end{cases}. \tag{1.10.9}$$

Comparing with (1.3.8) and noting that, there, we did not take into account transformations of the ξ^μ's, it is easy to realize that the sections of the gauge-natural (vector) bundle $W^{1,1}P \times_\lambda \mathcal{V}$ are in 1-1 correspondence with the G-invariant vector fields on P.

Example 1.10.19 (Bundle of vertical *G*-invariant vector fields). Take $\mathfrak{g}$ as the standard fibre and consider the following action:

$$\begin{cases} \lambda\colon W^{1,1}_m G \times \mathfrak{g} \to \mathfrak{g} \\ \lambda\colon \big((\alpha^j{}_k, a^{\mathcal{B}}, a^{\mathcal{C}}{}_l), w^{\mathcal{A}}\big) \mapsto (\mathrm{Ad}_a)^{\mathcal{A}}{}_{\mathcal{B}} w^{\mathcal{B}} \end{cases}. \tag{1.10.10}$$

Comparing with (1.3.9), it is immediate to realize that the sections of $W^{1,1}P \times_\lambda \mathfrak{g}$ are in 1-1 correspondence with the vertical G-invariant vector fields on P. Of course, in this example, we could more simply think of vertical G-invariant vector field as sections of the vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ associated with $P \cong W^{0,0}P$, i.e. of a gauge-natural vector bundle of order $(0, 0)$. Then, giving action (1.10.10) amounts to regarding the original G-manifold $\mathfrak{g}$ as a $W^{1,1}_m G$-manifold via the canonical projection of Lie groups $W^{1,1}_m G \to G$. It is also meaningful to think of action (1.10.10) as setting $v^i = 0$ in (1.10.9), and hence one sees that the first jet contribution, i.e. $a^{\mathcal{A}}{}_i$, disappears.

Chapter 2

General theory of Lie derivatives

Para ver una cosa hay que comprenderla.
J. L. Borges, *There are more things*

This chapter contains recent original results in the general theory of Lie derivatives, most of which were first presented in Godina & Matteucci (2002). This work was specifically motivated by the desire to gain a better understanding of the long-debated concept of a Lie derivative of spinor fields.

Indeed, it has now become apparent that there has been some confusion regarding this notion, both in the mathematical and the physical literature. Lichnerowicz was the first one to give a correct definition of a Lie derivative of spinor fields, although with respect to infinitesimal isometries only. The local expression given by Lichnerowicz (1963) is[1]

$$\mathcal{L}_\xi \psi := \xi^a \nabla_a \psi - \frac{1}{4} \nabla_a \xi_b \gamma^a \gamma^b \psi, \tag{$*$}$$

where $\nabla_a \xi_b = \nabla_{[a} \xi_{b]}$, as ξ is assumed to be a Killing vector field.

After a first attempt to extend Lichnerowicz's definition to generic infinitesimal transformations (Kosmann 1966), Kosmann put forward a new definition of a Lie derivative of spinor fields in her doctoral thesis under Lichnerowicz's supervision (1972). Indeed, in her previous work she had just extended *tout court* Lichnerowicz's definition to the case of a generic vector field ξ, without antisymmetrizing $\nabla_a \xi_b$. Therefore, the local expression appearing in Kosmann (1966) could not be given any clear-cut geometrical meaning. The remedy was then realized to be retaining Lichnerowicz's local expression ($*$) for a *generic* vector field ξ, but explicitly taking the antisymmetric part of $\nabla_a \xi_b$ only (Kosmann 1972).

Several papers on the subject followed, including particularly Binz & Pferschy's (1983) and Bourguignon & Gauduchon's (1992). Furthermore, among the physics community much interest has been attracted by Penrose & Rindler's (1986) definition, despite its being restricted to infinitesimal conformal isometries (see §2.7.1 below).

In this chapter we shall investigate whether the definition of a Lie derivative of spinor fields can be placed in the more general framework of the theory of Lie derivatives of sections of fibred manifolds (and, more generally, of differentiable maps between two manifolds) stemming from Trautman's (1972) seminal paper and further developed by Janyška & Kolář (1982).

[1]The reader is referred to Appendix D for some preliminaries on spinor theory and notation.

A first step in this direction was already taken by Fatibene *et al.* (1996), who successfully placed Kosmann's (1972) definition in the framework of the theory of Lie derivatives of sections of gauge-natural bundles by introducing a new geometric concept, which the authors called the "Kosmann lift".

The aim of this work is to provide a more transparent geometric explanation of the Kosmann lift and, at the same time, a generalization to reductive G-structures. Indeed, the Kosmann lift is but a *particular case* of this interesting generalization.

2.1 Generalized notion of a Lie derivative

Definition 2.1.1. Let N and N' be two manifolds and $f\colon N \to N'$ a map between them. By a ***vector field along*** f we shall mean a map $\zeta\colon N \to TN'$ such that $\tau_{N'} \circ \zeta = f$, $\tau_{N'}\colon TN' \to N'$ denoting the canonical tangent bundle projection.

Definition 2.1.2. Let N, N' and f be as above, and let η and η' be two vector fields on N and N', respectively. Then, by the ***generalized Lie derivative*** $\tilde{\mathcal{L}}_{(\eta,\eta')}f$ ***of*** f ***with respect to*** η ***and*** η' we shall mean the vector field along f given by

$$\tilde{\mathcal{L}}_{(\eta,\eta')}f := Tf \circ \eta - \eta' \circ f. \tag{2.1.1}$$

If $\{\varphi_t^\eta\}$ and $\{\varphi_t^{\eta'}\}$ denote the flows of η and η', respectively, then one readily verifies that

$$\tilde{\mathcal{L}}_{(\eta,\eta')}f = \frac{\partial}{\partial t}(\varphi_{-t}^{\eta'} \circ f \circ \varphi_t^\eta)\bigg|_{t=0}. \tag{2.1.1$'$}$$

Indeed, on performing the differentiation on the r.h.s. of (2.1.1′) and recalling the definition of a tangent map and a flow of a vector field (*cf.* §§1.1 and B.1), one gets

$$\begin{aligned}
\tfrac{\partial}{\partial t}(\varphi_{-t}^{\eta'} \circ f \circ \varphi_t^\eta)\Big|_{t=0}(x) &= \Big(\tfrac{\partial}{\partial t}\varphi_{-t}^{\eta'}\Big|_{t=0} \circ f\Big)(\varphi_0^\eta(x)) + \Big(T_{f(\varphi_0^\eta(x))}\varphi_0^{\eta'} \circ T_{\varphi_0^\eta(x)}f \circ \tfrac{\partial}{\partial t}\varphi_t^\eta\Big|_{t=0}\Big)(x)\\
&= -(\eta' \circ f)(\mathrm{id}_N(x)) + \Big(T_{f(\mathrm{id}_N(x))}\mathrm{id}_{N'} \circ T_{\mathrm{id}_N(x)}f \circ \eta\Big)(x)\\
&= \Big(\mathrm{id}_{T_{f(x)}N'} \circ T_x f \circ \eta\Big)(x) - (\eta' \circ f)(x)\\
&= (T_x f \circ \eta - \eta' \circ f)(x)
\end{aligned}$$

for all $x \in N$, i.e. (2.1.1). □

The concept of a generalized Lie derivative was first introduced by Trautman (1972) and further developed by Kolář (1982) and Janyška & Kolář (1982) (see also Kolář *et al.* 1993, Chapter XI).

Consider now two fibred manifolds (B, M, π) and (B', M, π'), a base-preserving morphism $\Phi\colon B \to B'$, and two projectable vector fields $\eta \in \mathfrak{X}(B)$ and $\eta' \in \mathfrak{X}(B')$ over the same vector field $\xi \in \mathfrak{X}(M)$. Then, from (2.1.1) and the fact that Φ is base-preserving,

$$\begin{aligned} T\pi' \circ \tilde{\pounds}_{(\eta,\eta')}\Phi &= T\pi' \circ (T\Phi \circ \eta - \eta' \circ \Phi) \\ &= T(\pi' \circ \Phi) \circ \eta - T\pi' \circ \eta' \circ \Phi \\ &= T(\mathrm{id}_M \circ \pi) \circ \eta - \xi \circ \pi' \circ \Phi \\ &= T\pi \circ \eta - \xi \circ \mathrm{id}_M \circ \pi \\ &= \xi \circ \pi - \xi \circ \pi = 0_{TM}. \end{aligned}$$

Therefore,

$$\tilde{\pounds}_{(\eta,\eta')}\Phi \colon B \to VB'. \tag{2.1.2}$$

Remark 2.1.3. One says that a fibred manifold (B, M, π) admits a *vertical splitting* if there exists a linear bundle isomorphism $\alpha_B \colon VB \to B \times_M \bar{B}$ (covering the identity of B), where $(\bar{B}, M, \bar{\pi})$ is a vector bundle. In particular, a vector bundle (E, M, π) admits a *canonical* vertical splitting $\alpha_E \colon VE \to E \times_M E$. Indeed, if $\check{\tau}_E \colon TE \to E$ denotes the (canonical) tangent bundle projection restricted to VE, y is a point in E such that $y = \check{\tau}_E(v)$ for a given $v \in VE$, and $\gamma \colon \mathbb{R} \to E_y \equiv \pi^{-1}(\pi(y))$ is a curve such that $\gamma(0) = y$ and $j^1_0\gamma = v$, then α_E is given by $\alpha_E(v) := (y, w)$, where $w := \lim_{t\to 0} \frac{1}{t}(\gamma(t) - \gamma(0))$. Analogously, an affine bundle (A, M, π) modelled on a vector bundle $(\vec{A}, M, \vec{\pi})$ admits a *canonical* vertical splitting $\alpha_A \colon VA \to A \times_M \vec{A}$, defined exactly as before with the caveat that now $\gamma(t) - \gamma(0) \in \vec{A}_y$ for all $t \in \mathbb{R}$.

So, if we specialize (2.1.2) to the case in which (B', M, π') admits a vertical splitting $\alpha_{B'} \colon VB' \to B' \times_M \bar{B}'$, we have that the second component of $\tilde{\pounds}_{(\eta,\eta')}$ is a map

$$\pounds_{(\eta,\eta')}\Phi \colon B \to \bar{B}', \tag{2.1.3}$$

which we shall call the (***restricted***) ***Lie derivative of*** Φ ***with respect to*** η ***and*** η'.

Now, let (B, M, π) a fibred manifold, $\Xi \in \mathfrak{X}(B)$ a projectable vector field over a vector field $\xi \in \mathfrak{X}(M)$, and $\sigma \colon M \to B$ a section of B. Then, from (2.1.1),

$$\begin{aligned} T\pi \circ \tilde{\pounds}_{(\xi,\Xi)}\sigma &= T\pi \circ (T\sigma \circ \xi - \Xi \circ \sigma) \\ &= T(\pi \circ \sigma) \circ \xi - T\pi \circ \Xi \circ \sigma \\ &= T\mathrm{id}_M \circ \xi - \xi \circ \pi \circ \sigma \\ &= \mathrm{id}_{TM} \circ \xi - \xi \circ \mathrm{id}_M \\ &= \xi - \xi = 0_{TM}. \end{aligned}$$

Therefore,

$$\tilde{\pounds}_{\Xi}\sigma := \tilde{\pounds}_{(\xi,\Xi)}\sigma \colon M \to VB. \tag{2.1.4}$$

$\tilde{\pounds}_{\Xi}\sigma$ is called the ***generalized Lie derivative of*** σ ***with respect to*** Ξ. If (B, M, π) admits a vertical splitting $\alpha_B \colon VB \to B \times_M \bar{B}$, then, as before, we shall call the second component

$$\pounds_{\Xi}\sigma \colon M \to \bar{B} \tag{2.1.5}$$

of $\tilde{\pounds}_{\Xi}\sigma$ the (***restricted***) ***Lie derivative of*** σ ***with respect to*** Ξ. In this case, if $(x^\lambda, y^{\mathfrak{a}})$ are local fibred coordinates on B and $\sigma^{\mathfrak{a}} := y^{\mathfrak{a}} \circ \sigma$, by virtue of (2.1.1) we can locally write[2]

$$\pounds_\Xi \sigma = (\xi^\mu \partial_\mu \sigma^\mathfrak{a} - \Xi^\mathfrak{a} \circ \sigma) \frac{\partial}{\partial y^\mathfrak{a}}, \tag{2.1.6}$$

$\xi^\mu \partial_\mu$ and $\xi^\mu \partial_\mu + \Xi^\mathfrak{a} \partial_\mathfrak{a}$ being the local expressions of ξ and Ξ, respectively. Also, on using the fact that the second component of $\tilde{\pounds}_\Xi \sigma$ is the derivative of $\Phi_{-t} \circ \sigma \circ \varphi_t$ at $t = 0$ in the classical sense, one can re-express the restricted Lie derivative in the form

$$\pounds_\Xi \sigma = \lim_{t \to 0} \frac{\Phi_{-t} \circ \sigma \circ \varphi_t - \sigma}{t}. \tag{2.1.5$'$}$$

An important property of $\tilde{\pounds}_\Xi \sigma$ is that it commutes with j^k. Indeed, if $\{\varphi_t\}$ and $\{\Phi_t\}$ denote the flows of ξ and Ξ, respectively, from (2.1.1$'$) and (1.7.2) it follows that

$$\begin{aligned} j^k \tilde{\pounds}_\Xi \sigma &= \tfrac{\partial}{\partial t} j^k (\Phi_{-t} \circ \sigma \circ \varphi_t)\Big|_{t=0} \\ &= \tfrac{\partial}{\partial t} (J^k \Phi_{-t} \circ j^k \sigma \circ \varphi_t)\Big|_{t=0} \\ &\equiv \tilde{\pounds}_{J^k \Xi}\, j^k \sigma =: \tilde{\pounds}_\Xi\, j^k \sigma, \end{aligned}$$

as claimed. In local coordinates this reads of course[3]

$$\partial_{\boldsymbol{\mu}} \tilde{\pounds}_\Xi \sigma^\mathfrak{a} = \tilde{\pounds}_\Xi \partial_{\boldsymbol{\mu}} \sigma^\mathfrak{a}, \tag{2.1.7}$$

$1 \leqslant |\boldsymbol{\mu}| \leqslant k$. □

We can now specialize formula (2.1.4) to the case of gauge-natural bundles in a straightforward manner.

Definition 2.1.4. Let P_λ be a gauge-natural bundle associated with some principal bundle $P(M, G)$, Ξ a G-invariant vector field on P projecting over a vector field ξ on M, and $\sigma\colon M \to P_\lambda$ a section of P_λ. Then, by the ***generalized*** (***gauge-natural***) ***Lie derivative of*** σ ***with respect to*** Ξ we shall mean the map

$$\tilde{\pounds}_\Xi \sigma\colon M \to VP_\lambda, \quad \tilde{\pounds}_\Xi \sigma := T\sigma \circ \xi - \Xi_\lambda \circ \sigma, \tag{2.1.8}$$

where Ξ_λ is the gauge-natural lift of Ξ onto P_λ as per Definition 1.10.12. Equivalently,

$$\tilde{\pounds}_\Xi \sigma = \frac{\partial}{\partial t} \Big((\Phi_{-t})_\lambda \circ \sigma \circ \varphi_t \Big)\Big|_{t=0}, \tag{2.1.8$'$}$$

$\{\varphi_t\}$ and $\{\Phi_t\}$ denoting the flows of ξ and Ξ, respectively.

In particular, if $\mathscr{F}M$ is a natural bundle over M and $\mathscr{F}\xi \in \mathfrak{X}(\mathscr{F}M)$ denotes the

[2] Of course, the r.h.s. of (2.1.6) is also the local expression of $\tilde{\pounds}_\Xi \sigma$ in the general case, but, as such, it does not define a global object unless (B, M, π) admits a vertical splitting, in which case the global object in question is precisely $\pounds_\Xi \sigma$.

[3] Identity (2.1.7) might look surprising at first sight, but one can convince oneself that it is indeed the right answer by considering that the only sensible way to take the Lie derivative of the k-th partial derivative of a section of some fibre bundle is to regard such partial derivative as the local expression of the k-th order jet prolongation of the given section, whence (2.1.7) *naturally* follows.

natural lift of a vector field $\xi \in \mathfrak{X}(M)$, we shall write

$$\tilde{\pounds}_\xi \sigma := \tilde{\pounds}_{\mathscr{F}\xi}\sigma \colon M \to V\mathscr{F}M, \tag{2.1.9}$$

which will be simply called the ***generalized Lie derivative of*** σ ***with respect to*** ξ. Of course, if P_λ [$\mathscr{F}M$] admits a vertical splitting, we can define the notion of a (restricted) Lie derivative corresponding to (2.1.8) [(2.1.9)] in the usual fashion. In particular, on specializing (2.1.9) to the case of tensor (density) bundles over M, one recovers the standard definition of a Lie derivative given in classical textbooks (*cf.*, e.g, Schouten 1954; Yano 1957). For instance, in the case of the tangent [cotangent] bundle, applying (2.1.6) with $\Xi = T\xi$ [$\Xi = T^*\xi$] gives [*cf.* (1.6.4)]

$$\pounds_\xi \eta^\nu = \xi^\mu \partial_\mu \eta^\nu - \eta^\mu \partial_\mu \xi^\nu \equiv [\xi, \eta]^\nu \qquad [\pounds_\xi \alpha_\nu = \xi^\mu \partial_\mu \alpha_\nu + \alpha_\mu \partial_\nu \xi^\mu],$$

which is indeed the usual coordinate expression for the Lie derivative of a vector field η with $\eta^\nu := y^\nu \circ \eta$ [1-form α with $\alpha_\nu := y_\nu \circ \alpha$]. Alternatively, using (2.1.5′), we find

$$\pounds_\xi \eta = \lim_{t\to 0} \frac{T\varphi_{-t} \circ \eta \circ \varphi_t - \eta}{t} \equiv \lim_{t\to 0} \frac{\varphi_t^* \eta - \eta}{t}$$
$$\left[\pounds_\xi \alpha = \lim_{t\to 0} \frac{(T\varphi_t)^* \circ \alpha \circ \varphi_t - \alpha}{t} \equiv \lim_{t\to 0} \frac{\varphi_t^* \alpha - \alpha}{t}\right],$$

which is the corresponding classical intrinsic expression.

Remark 2.1.5. We stress that the concept of a (generalized or restricted) gauge-natural Lie derivative is, crucially, a *category-dependent* one. In confirmation of this, recall that in Example 1.10.15 we mentioned the fact that we have (at least) three different notions of a Lie derivative of a vector field $\eta \in \mathfrak{X}(M)$, depending on whether we regard TM as natural bundle (associated with LM), a gauge-natural bundle associated with LM, or a gauge-natural bundle associated with $\mathrm{SO}(M, g)$.[4] We have just obtained both the coordinate and the intrinsic expression for the Lie derivative of η corresponding to the first case. As for the other two, note for a start that, unlike in the first case, we cannot take the Lie derivative of η with respect to another vector field ξ on M, but we can only take the Lie derivative of η with respect to a $\mathrm{GL}(m, \mathbb{R})$-invariant [$\mathrm{SO}(p, q)^e$-invariant] vector field Ξ on LM [$\mathrm{SO}(M, g)$] projecting on ξ. This is because the bundles under consideration are *not* natural, and we cannot functorially lift ξ onto them, but only Ξ. Then, on applying Definition 2.1.4 and taking (1.10.6), (1.4.2) [(1.4.4)] and (1.3.10) into account, we find [see also (2.2.1) below]

$$\pounds_\Xi \eta^a = \xi^\mu \partial_\mu \eta^a - \Xi^a{}_b \eta^b, \tag{2.1.10}$$

where $\|\Xi^a{}_b(x)\| \in \mathfrak{gl}(m, \mathbb{R})$ [$\|\Xi^a{}_b(x)\| \in \mathfrak{so}(p, q)$]. Of course, on using (1.3.14) we could re-express (2.1.10) in holonomic coordinates, but this would not change the fact that $\|\Xi^a{}_b(x)\|$ *is a generic element of* $\mathfrak{gl}(m, \mathbb{R})$ [$\mathfrak{so}(p, q)$], a priori *unrelated to the* $\xi^\mu(x)$*'s.*[5]

[4] Note that the three bundles under consideration are isomorphic as *vector bundles*, *not* as *functors*.

[5] In order to re-express (2.1.10) in holonomic coordinates in the case $TM \cong \mathrm{SO}(M, g) \times_{\lambda'} \mathbb{R}^m$ using a relation formally identical with (1.3.14), we need the concept of an $\mathrm{SO}(p, q)^e$-tetrad, which shall be

In the sequel we shall also need the following

Definition 2.1.6. We call ***formal generalized Lie derivative*** the (global) base-preserving morphism (over M) $\tilde{\pounds}_\Xi y\colon J^1B \to VB$ intrinsically defined by

$$\tilde{\pounds}_\Xi y \circ j^1\sigma = \tilde{\pounds}_\Xi \sigma, \tag{2.1.11}$$

where $\tilde{\pounds}_\Xi \sigma$ is given by (2.1.8). Locally,

$$\tilde{\pounds}_\Xi y = (\xi^\mu y^\mathfrak{a}{}_\mu - \Xi^\mathfrak{a})\frac{\partial}{\partial y^\mathfrak{a}}. \tag{2.1.11$'$}$$

Thus, the formal generalized Lie derivative $\tilde{\pounds}_\Xi y$ is but the generalized Lie derivative operator $\tilde{\pounds}_\Xi$ *regarded* as a fibred morphism.

There is an important relation between vertical differential and formal (generalized) Lie derivative. Indeed,

$$\begin{aligned} J^1\Xi \,\lrcorner\, \mathrm{d_V} y^\mathfrak{a} &\equiv J^1\Xi \,\lrcorner\, (\mathrm{d}y^\mathfrak{a} - y^\mathfrak{a}{}_\mu\, \mathrm{d}x^\mu) \\ &= \Xi^\mathfrak{a} - \xi^\mu y^\mathfrak{a}{}_\mu \\ &= -\tilde{\pounds}_\Xi y^\mathfrak{a}, \end{aligned} \tag{2.1.12}$$

$J^1\Xi$ being given by (1.9.3) with $\Xi^\mu = \xi^\mu$. Another important property of the formal Lie derivative is that it commutes with the formal derivative. Indeed, on applying (2.1.11$'$) to $y^\mathfrak{a}{}_\mu$, we obtain

$$\begin{aligned} \tilde{\pounds}_\Xi y^\mathfrak{a}{}_\mu \equiv \tilde{\pounds}_{J^1\Xi} y^\mathfrak{a}{}_\mu &= \xi^\nu y^\mathfrak{a}{}_{\mu\nu} - \Xi^\mathfrak{a}{}_\mu \\ &= \xi^\nu y^\mathfrak{a}{}_{\mu\nu} - (\mathrm{d}_\mu \Xi^\mathfrak{a} - y^\mathfrak{a}{}_\nu \partial_\mu \xi^\nu), \end{aligned} \tag{2.1.13}$$

where we used (1.9.3) with $\Xi^\mu = \xi^\mu$. On the other hand, taking the formal derivative of (2.1.11$'$) gives

$$\begin{aligned} \mathrm{d}_\mu \tilde{\pounds}_\Xi y^\mathfrak{a} &= \mathrm{d}_\mu(\xi^\nu y^\mathfrak{a}{}_\nu - \Xi^\mathfrak{a}) \\ &= y^\mathfrak{a}{}_\nu \mathrm{d}_\mu \xi^\nu + \xi^\nu y^\mathfrak{a}{}_{\nu\mu} - \mathrm{d}_\mu \Xi^\mathfrak{a} \\ &= \xi^\nu y^\mathfrak{a}{}_{\mu\nu} - (\mathrm{d}_\mu \Xi^\mathfrak{a} - y^\mathfrak{a}{}_\nu \partial_\mu \xi^\nu) = \tilde{\pounds}_\Xi y^\mathfrak{a}{}_\mu, \end{aligned} \tag{2.1.14}$$

as claimed.[6] □

2.2 Lie derivatives and Lie algebras

Definition 2.2.1. Let E be a vector bundle over a manifold M. Then, TE is a vector bundle over TM. A projectable vector field $\Xi \in \mathfrak{X}(E)$ over a vector field $\xi \in \mathfrak{X}(M)$ is called a ***linear vector field*** if $\Xi\colon E \to TE$ is a linear morphism of E into TE over the

introduced in §2.6 below.

[6] Of course, this could also be deduced directly from (2.1.7), (1.8.16) and (2.1.11).

base map $\xi\colon M \to TM$. Locally,

$$\Xi(x,y) = \xi^\mu(x)\partial_\mu + \Xi^a{}_b(x)y^b\partial_a \tag{2.2.1}$$

for all $\psi_\alpha^{-1}(x,y) \in E$.

Analogously, we can give

Definition 2.2.2. Let A be an affine bundle over a manifold M. Then, TA is an affine bundle over TM. A projectable vector field $\tilde{\Xi} \in \mathfrak{X}(A)$ over a vector field $\xi \in \mathfrak{X}(M)$ is called an ***affine vector field*** if $\tilde{\Xi}\colon A \to TA$ is an affine morphism of A into TA over the base map $\xi\colon M \to TM$. Locally[7],

$$\tilde{\Xi}(x,y) = \xi^\mu(x)\partial_\mu + (\Xi^a{}_b(x)y^b + \Xi^a(x))\partial_a \equiv \Xi(x,y) + \Xi^a(x)\partial_a \tag{2.2.2}$$

for all $\psi_\alpha^{-1}(x,y) \in A$.

Now, let Ξ and H be two linear vector fields on a vector bundle E over a manifold M. Then, using (2.2.1) (and the analogous one for H), we find

$$[\Xi, \mathrm{H}] = [\xi,\eta]^\mu\partial_\mu + (\xi^\mu\partial_\mu \mathrm{H}^a{}_b - \eta^\mu\partial_\mu\Xi^a{}_b + \Xi^c{}_b\mathrm{H}^a{}_c - \mathrm{H}^c{}_b\Xi^a{}_c)y^b\partial_a,$$

whence of course

$$\pounds_{[\Xi,\mathrm{H}]}\sigma^a = [\xi,\eta]^\mu\partial_\mu\sigma^a - (\xi^\mu\partial_\mu \mathrm{H}^a{}_b - \eta^\mu\partial_\mu\Xi^a{}_b + \Xi^c{}_b\mathrm{H}^a{}_c - \mathrm{H}^c{}_b\Xi^a{}_c)\sigma^b.$$

On the other hand,

$$\begin{aligned}[\pounds_\Xi, \pounds_\mathrm{H}]\sigma^a &= (\xi^\nu\partial_\nu\eta^\mu\partial_\mu\sigma^a + \xi^\nu\eta^\mu\partial_\nu\partial_\mu\sigma^a - \xi^\mu\partial_\mu\mathrm{H}^a{}_b\sigma^b - \xi^\mu\mathrm{H}^a{}_b\partial_\mu\sigma^b - \eta^\mu\Xi^a{}_b\partial_\mu\sigma^b \\ &\quad + \mathrm{H}^c{}_b\Xi^a{}_c\sigma^b) - (\Xi \leftrightarrow \mathrm{H}) \\ &= \pounds_{[\Xi,\mathrm{H}]}\sigma^a.\end{aligned} \tag{2.2.3}$$

Similarly, let $\tilde{\Xi}$ and $\tilde{\mathrm{H}}$ be two affine vector fields on an affine bundle A over M. Then, using (2.2.2) (and the analogous one for $\tilde{\mathrm{H}}$), we find

$$[\tilde{\Xi}, \tilde{\mathrm{H}}] = [\Xi, \mathrm{H}] + (\xi^\mu\partial_\mu\mathrm{H}^a - \eta^\mu\partial_\mu\Xi^a + \Xi^b\mathrm{H}^a{}_b - \mathrm{H}^b\Xi^a{}_b)\partial_a,$$

whence

$$\pounds_{[\tilde{\Xi},\tilde{\mathrm{H}}]}\sigma^a = \pounds_{[\Xi,\mathrm{H}]}\sigma^a - (\xi^\mu\partial_\mu\mathrm{H}^a - \eta^\mu\partial_\mu\Xi^a + \Xi^b\mathrm{H}^a{}_b - \mathrm{H}^b\Xi^a{}_b).$$

On the other hand, if we define (consistently) $[\pounds_{\tilde{\Xi}}, \pounds_{\tilde{\mathrm{H}}}]$ to equal $\pounds_\Xi\pounds_{\tilde{\mathrm{H}}} - \pounds_\mathrm{H}\pounds_{\tilde{\Xi}}$,

$$\begin{aligned}[\pounds_{\tilde{\Xi}}, \pounds_{\tilde{\mathrm{H}}}]\sigma^a &= (\pounds_\Xi\pounds_\mathrm{H}\sigma^a - \pounds_\Xi\mathrm{H}^a) - (\tilde{\Xi} \leftrightarrow \tilde{\mathrm{H}}) \\ &= (\pounds_\Xi\pounds_\mathrm{H}\sigma^a - \xi^\mu\partial_\mu\mathrm{H}^a + \Xi^a{}_b\mathrm{H}^b) - (\tilde{\Xi} \leftrightarrow \tilde{\mathrm{H}}) \\ &= \pounds_{[\tilde{\Xi},\tilde{\mathrm{H}}]}\sigma^a.\end{aligned} \tag{2.2.4}$$

[7]For consistency with (2.2.1), in (2.2.2) and the rest of this section (y^a) will always denote fibre coordinates on the vector bundle E, on which A is assumed to be modelled, rather than on A itself.

Now, by its very definition, every gauge-natural lift Ξ_λ of a G-invariant vector field Ξ on a principal bundle $P(M,G)$ onto an associated gauge-natural vector [affine] bundle P_λ over M is a linear [affine] vector field [*cf.*, e.g., (2.1.10)]. Therefore,

$$[\pounds_\Xi, \pounds_\mathrm{H}] \equiv [\pounds_{\Xi_\lambda}, \pounds_{\mathrm{H}_\lambda}] = \pounds_{[\Xi_\lambda, \mathrm{H}_\lambda]} = \pounds_{[\Xi, \mathrm{H}]_\lambda} \equiv \pounds_{[\Xi, \mathrm{H}]}, \tag{2.2.5}$$

where the second identity follows from (2.2.3) [(2.2.4)] and the third one from Proposition 1.10.14. Hence, $\pounds$ is a Lie algebra homomorphism from $\mathfrak{X}_G(P)$ to $\mathrm{End}C^\infty(P_\lambda)$.

Furthermore, if $\mathscr{F}\xi$ denotes the natural lift of a vector field ξ on M onto a natural vector [affine] bundle $\mathscr{F}M$, then

$$[\pounds_\xi, \pounds_\eta] \equiv [\pounds_{\mathscr{F}\xi}, \pounds_{\mathscr{F}\eta}] = \pounds_{[\mathscr{F}\xi, \mathscr{F}\eta]} = \pounds_{\mathscr{F}[\xi,\eta]} \equiv \pounds_{[\xi,\eta]}, \tag{2.2.6}$$

where the second identity follows from (2.2.5) and the third one from Proposition 1.6.7. Hence, $\pounds$ is here a Lie algebra homomorphism from $\mathfrak{X}(M)$ to $\mathrm{End}C^\infty(\mathscr{F}M)$.

Finally, it should be mentioned that (2.2.5) and (2.2.6) can be derived from a very general formula first proven by Kolář (1982) (see also Kolář *et al.* 1993, §50). Our main aim here was to show that a "bracket formula" holds for (gauge-) natural Lie derivatives in the context of (gauge-) natural vector and affine bundles, which constitute the geometrical arena of classical field theory.

2.3 Reductive G-structures and their prolongations

Definition 2.3.1. Let H be a Lie group and G a Lie subgroup of H. Denote by $\mathfrak{h}$ the Lie algebra of H and by $\mathfrak{g}$ the Lie algebra of G. We shall say that G is a ***reductive Lie subgroup*** of H if there exists a direct sum decomposition

$$\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{m},$$

where $\mathfrak{m}$ is an Ad_G-invariant vector subspace of $\mathfrak{h}$, i.e. $\mathrm{Ad}_a\mathfrak{m} \subset \mathfrak{m}$ for all $a \in G$.

Remark 2.3.2. A Lie algebra $\mathfrak{h}$ and a Lie subalgebra $\mathfrak{g}$ satisfying these properties form a so-called *reductive pair* (*cf.* Choquet-Bruhat & DeWitt-Morette 1989, p. 103). Moreover, $\mathrm{Ad}_G\mathfrak{m} \subset \mathfrak{m}$ implies $(T_e\mathrm{Ad})_\mathfrak{g}\mathfrak{m} \equiv \mathrm{ad}_\mathfrak{g}\mathfrak{m} \equiv [\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{m}$, and, conversely, if G is connected, $[\mathfrak{g}, \mathfrak{m}] \subset \mathfrak{m}$ implies $\mathrm{Ad}_G\mathfrak{m} \subset \mathfrak{m}$ (*cf.* §C.2).

Example 2.3.3. Consider a subgroup $G \subset H$ and suppose that an Ad_G-invariant metric K can be assigned on the Lie algebra $\mathfrak{h}$ (e.g., if H is a semisimple Lie group, K could be the *Killing-Cartan form*, given by $K(\xi_e, \eta_e) = \mathrm{tr}(\mathrm{ad}_{\xi_e} \circ \mathrm{ad}_{\eta_e})$ for all $\xi_e, \eta_e \in \mathfrak{h}$: indeed, this form is Ad_H-invariant and, in particular, also Ad_G-invariant). Set

$$\mathfrak{m} := \mathfrak{g}^\perp \equiv \{\, \xi_e \in \mathfrak{h} \mid K(\xi_e, \eta_e) = 0 \ \forall \eta_e \in \mathfrak{g} \,\}.$$

Obviously, $\mathfrak{h}$ can be decomposed as the direct sum $\mathfrak{h} = \mathfrak{g} \oplus \mathfrak{m}$ and it is easy to show that, under the assumption of Ad_G-invariance of K, the vector subspace $\mathfrak{m}$ is also Ad_G-invariant.

Example 2.3.4 (The unimodular group). The unimodular group $\mathrm{SL}(m,\mathbb{R})$ is an example of a reductive Lie subgroup of $\mathrm{GL}(m,\mathbb{R})$. To see this, first recall that its Lie algebra $\mathfrak{sl}(m,\mathbb{R})$ is formed by all $m\times m$ traceless matrices (*cf.* §C.2). If M is any matrix in $\mathfrak{gl}(m,\mathbb{R})$, the following decomposition holds:

$$\mathsf{M} = \mathsf{U} + \frac{1}{m}\,\mathrm{tr}(\mathsf{M})\mathsf{I},$$

where $\mathsf{I} := \mathrm{id}_{\mathfrak{gl}(m,\mathbb{R})}$ and U is traceless. Indeed,

$$\mathrm{tr}(\mathsf{U}) = \mathrm{tr}(\mathsf{M}) - \frac{1}{m}\,\mathrm{tr}(\mathsf{M})\,\mathrm{tr}(\mathsf{I}) = 0.$$

Accordingly, the Lie algebra $\mathfrak{gl}(m,\mathbb{R})$ can be decomposed as follows:

$$\mathfrak{gl}(m,\mathbb{R}) = \mathfrak{sl}(m,\mathbb{R}) \oplus \mathbb{R}.$$

In this case, $\mathfrak{m}$ is the set $\mathbb{R}\mathsf{I} \cong \mathbb{R}$ of all real multiples of I, which is obviously adjoint-invariant under $\mathrm{SL}(m,\mathbb{R})$. Indeed, if S is an arbitrary element of $\mathrm{SL}(m,\mathbb{R})$, for any $a\in\mathbb{R}$ one has

$$\mathrm{Ad}_{\mathsf{S}}(a\mathsf{I}) \equiv \mathsf{S}(a\mathsf{I})\mathsf{S}^{-1} = a\mathsf{I}\mathsf{S}\mathsf{S}^{-1} = a\mathsf{I}.$$

This proves that $\mathbb{R}$ is adjoint-invariant under $\mathrm{SL}(m,\mathbb{R})$, and $\mathrm{SL}(m,\mathbb{R})$ is a reductive Lie subgroup of $\mathrm{GL}(m,\mathbb{R})$. □

Given the importance of the following example for the future developments of the theory, we shall state it as

Proposition 2.3.5. *The (pseudo-) orthogonal group* $\mathrm{SO}(p,q)$, $p+q=m$, *is a reductive Lie subgroup of* $\mathrm{GL}(m,\mathbb{R})$.

Proof. Let η denote the standard metric of signature (p,q), $p+q=m$, on $\mathbb{R}^m$ (*cf.* §C.1) and M be any matrix in $\mathfrak{gl}(m,\mathbb{R})$. Denote by $\mathsf{M}^\top$ the adjoint ("transpose") of M with respect to η, defined by requiring $\eta(\mathsf{M}^\top v, v') = \eta(v,\mathsf{M}v')$ for all $v,v'\in\mathbb{R}^m$. Of course, any traceless matrix can be (uniquely) written as the sum of an antisymmetric matrix and a symmetric traceless matrix. Therefore,

$$\mathfrak{sl}(m,\mathbb{R}) = \mathfrak{so}(p,q) \oplus \mathcal{V},$$

$\mathfrak{so}(p,q)$ denoting the Lie algebra of the (pseudo-) orthogonal group $\mathrm{SO}(p,q)$ for η, formed by all matrices A in $\mathfrak{gl}(m,\mathbb{R})$ such that $\mathsf{A}^\top = -\mathsf{A}$, and $\mathcal{V}$ the vector space of all matrices V in $\mathfrak{sl}(m,\mathbb{R})$ such that $\mathsf{V}^\top = \mathsf{V}$. Now, let O be any element of $\mathrm{SO}(p,q)$ and set $\mathsf{V}' := \mathrm{Ad}_{\mathsf{O}}\mathsf{V} \equiv \mathsf{O}\mathsf{V}\mathsf{O}^{-1}$ for any $\mathsf{V}\in\mathcal{V}$. We have

$$\mathsf{V}'^\top = (\mathsf{O}\mathsf{V}\mathsf{O}^\top)^\top = \mathsf{V}'$$

because $\mathsf{V}^\top = \mathsf{V}$ and $\mathsf{O}^{-1} = \mathsf{O}^\top$. Moreover,

$$\mathrm{tr}(\mathsf{V}') = \mathrm{tr}(\mathsf{O})\,\mathrm{tr}(\mathsf{V})\,\mathrm{tr}(\mathsf{O}^{-1}) = 0$$

since V is traceless. So, V' is in $\mathcal{V}$, thereby proving that $\mathcal{V}$ is adjoint-invariant under

$\mathrm{SO}(p, q)$. Therefore, $\mathrm{SO}(p, q)$ is a reductive Lie subgroup of $\mathrm{SL}(m, \mathbb{R})$ and, hence, also a reductive Lie subgroup of $\mathrm{GL}(m, \mathbb{R})$ by virtue of Example 2.3.4. □

Definition 2.3.6. A ***reductive*** G***-structure*** on a principal bundle $Q(M, H)$ is a principal subbundle $P(M, G)$ of $Q(M, H)$ such that G is a reductive Lie subgroup of H.

Now, since later on we shall consider the case of spinor fields, it is convenient to give the following general

Definition 2.3.7. Let $P(M, G)$ be a principal bundle and $\rho\colon \Gamma \to G$ a central homomorphism of a Lie group Γ onto G, i.e. such that its kernel is discrete[8] and contained in the centre of Γ (Greub & Petry 1978; Haefliger 1956). A Γ***-structure*** on $P(M, G)$ is a principal bundle map $\zeta\colon \tilde{P} \to P$ which is equivariant under the right actions of the structure groups, i.e.

$$\zeta(\tilde{u} \cdot \alpha) = \zeta(\tilde{u}) \cdot \rho(\alpha)$$

for all $\tilde{u} \in \tilde{P}$ and $\alpha \in \Gamma$.

Equivalently, we have the following commutative diagrams

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\zeta} & P \\ {\scriptstyle\tilde{\pi}}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & \xrightarrow[\mathrm{id}_M]{} & M \end{array} \qquad\qquad \begin{array}{ccc} \tilde{P} & \xrightarrow{\tilde{R}_\alpha} & \tilde{P} \\ {\scriptstyle\zeta}\downarrow & & \downarrow{\scriptstyle\zeta} \\ P & \xrightarrow[\tilde{R}_a]{} & P \end{array}$$

$\tilde{R}_a$ and $\tilde{R}_\alpha$ denoting the canonical right actions on P and $\tilde{P}$, respectively. This means that for $\tilde{u} \in \tilde{P}$, both $\tilde{u}$ and $\zeta(\tilde{u})$ lie over the same point, and ζ, restricted to any fibre, is a "copy" of ρ, i.e. it is equivalent to it. The existence condition for a Γ-structure can be formulated in terms of Čech cohomology (*cf.* Haefliger 1956; Greub & Petry 1978; Lawson & Michelsohn 1989).

Remark 2.3.8. Recall that, if X and Y are two topological spaces and X is arcwise connected and arcwise locally connected, a map $f\colon Y \to X$ is called a *covering* if $f(Y) = X$ and, for each $x \in X$, there is an arcwise connected neighbourhood U of x such that each component of $f^{-1}(U)$ is open in Y and maps topologically onto U under f. The space Y is then called a *covering space*. It can be shown that a covering space Y admits a fibre bundle structure with a discrete structure group (*cf.* Steenrod 1951, §14.3). Then, the bundle map $\zeta\colon \tilde{P} \to P$ is a covering, and $\tilde{P}$ a covering space.

Proposition 2.3.9. *Let $\zeta\colon \tilde{P} \to P$ be a Γ-structure on $P(M, G)$. Then, every G-invariant vector field Ξ on P admits a unique (Γ-invariant) lift $\tilde{\Xi}$ onto $\tilde{P}$.*

Proof. Consider a G-invariant vector field Ξ, its flow being denoted by $\{\Phi_t\}$. For each $t \in \mathbb{R}$, Φ_t is an automorphism of P. Moreover, $\zeta\colon \tilde{P} \to P$ being a covering space, it is possible to lift Φ_t to a (unique) bundle map $\tilde{\Phi}_t\colon \tilde{P} \to \tilde{P}$ in the following way. For any point $\tilde{u} \in \tilde{P}$, consider the (unique) point $\zeta(\tilde{u}) = u$. From the theory of covering spaces

[8] We recall that a group G is called *discrete* if every subset of G is open, e.g. a finite group.

it follows that, for the curve $\gamma_u \colon \mathbb{R} \to P$ based at u, that is $\gamma_u(0) = u$, and defined by $\gamma_u(t) := \Phi_t(u)$, there exists a unique curve $\tilde{\gamma}_{\tilde{u}} \colon \mathbb{R} \to \tilde{P}$ based at $\tilde{u}$ such that $\zeta \circ \tilde{\gamma}_{\tilde{u}} = \gamma_u$ (*cf.* Steenrod 1951, §14). It is possible to define a principal bundle map $\tilde{\Phi}_t \colon \tilde{P} \to \tilde{P}$ covering Φ_t by setting $\tilde{\Phi}_t(\tilde{u}) := \tilde{\gamma}_{\tilde{u}}(t)$. The one-parameter group of automorphisms $\{\tilde{\Phi}_t\}$ of $\tilde{P}$ defines a vector field $\tilde{\Xi}(\tilde{u}) := \frac{\partial}{\partial t}[\tilde{\Phi}_t(\tilde{u})]\Big|_{t=0}$ for all $\tilde{u} \in \tilde{P}$. □

Proposition 2.3.10. *Let $\zeta \colon \tilde{P} \to P$ be a Γ-structure on $P(M,G)$. Then, every Γ-invariant vector field $\tilde{\Xi}$ on $\tilde{P}$ is projectable over a unique G-invariant vector field Ξ on P.*

Proof. Consider a Γ-invariant vector field $\tilde{\Xi}$ on $\tilde{P}$. Denote its flow by $\{\tilde{\Phi}_t\}$. Each $\tilde{\Phi}_t$ induces a unique automorphism $\Phi_t \colon P \to P$ such that $\zeta \circ \tilde{\Phi}_t = \Phi_t \circ \zeta$ and, hence, a unique vector field Ξ on P given by $\Xi(u) := \frac{\partial}{\partial t}[\Phi_t(u)]\Big|_{t=0}$ for all $u \in P$. □

Corollary 2.3.11. *Let $\zeta \colon \tilde{P} \to P$ be a Γ-structure on $P(M,G)$. There is a bijection between G-invariant vector fields on P and Γ-invariant vector fields on $\tilde{P}$.*

2.4 Split structures on principal bundles

Recall that a principal connection on a principal bundle $P(M,G)$ induces a decomposition $TP = HP \oplus VP$ of the tangent bundle (*cf.* §1.5). This is, of course, a well-known example of a "split structure" on a principal bundle. We shall now give the following general definition, due—for pseudo-Riemannian manifolds—to a number of authors (Walker 1955, 1958; Cattaneo-Gasparini 1963; Gray 1967; Fava 1968) and more generally to Gladush & Konoplya (1999).

Definition 2.4.1 (Godina & Matteucci 2002). An r***-split structure*** on a principal bundle $P(M,G)$ is a system of r fibre G-equivariant linear operators $\{\chi^i \colon TP \to TP\}_{i=1}^r$ of constant rank, equivalently viewed as 1-forms $\{\chi^i \in \Omega^1(P,TP)\}_{i=1}^r$, with the properties:

$$\chi^i \circ \chi^j = \delta^{ij}\chi^j, \qquad \sum_{i=1}^r \chi^i = \mathrm{id}_{TP}. \tag{2.4.1}$$

We introduce the notations:

$$\Sigma_u^i := \operatorname{im} \chi_u^i, \qquad n_i := \dim \Sigma_u^i, \tag{2.4.2}$$

where $\operatorname{im} \chi_u^i$ is the image of the operator χ^i at a point u of P, i.e. $\Sigma_u^i = \{\, v \in T_uP \mid \chi_u^i \circ v = v \,\}$. Owing to the constancy of the rank of the operators $\{\chi^i\}$, the numbers $\{n_i\}$ do not depend on the point u of P. It follows from the very definition of an r-split structure that we have a G-equivariant decomposition of the tangent space:

$$T_uP = \bigoplus_{i=1}^r \Sigma_u^i, \qquad \dim T_uP = \sum_{i=1}^r n_i.$$

Obviously, the bundle TP is also decomposed into r vector subbundles $\{\Sigma^i\}$ so that

$$TP = \bigoplus_{i=1}^r \Sigma^i, \qquad \Sigma^i = \bigcup_{u \in P} \Sigma_u^i. \tag{2.4.3}$$

Remark 2.4.2. Let M be a manifold. Recall that, given a vector subspace E_x of T_xM for each $x \in M$, the disjoint union $E := \coprod_{x\in M} E_x$ is called a *distribution* on M. Let $\mathfrak{X}_E(M)$ denote the set of all locally defined vector fields ξ on M such that $\xi(x) \in E_x$ whenever defined. Then, we say that E is a *smooth distribution* if $\mathfrak{X}_E(M)$ spans E. Finally, an *integral manifold* of a smooth distribution E is a connected submanifold N of M such that $T_xN = E_x$ for all $x \in N$, and E is called *integrable* if each point of M is contained in some integral manifold of E. Now, in general, the r vector subbundles $\{\Sigma^i \to P\}$ defined above are *anholonomic*, i.e. non-integrable, and are not vector subbundles of VP. For a principal connection, i.e. for the case $TP = HP \oplus VP$, the vector bundle VP is integrable.

Proposition 2.4.3. *An equivariant decomposition of TP into r vector subbundles $\{\Sigma^i\}$ as given by* (2.4.3)*, with $T_u\tilde{R}_a(\Sigma^i_u) = \Sigma^i_{u\cdot a}$, induces a system of r fibre G-equivariant linear operators $\{\chi^i \colon TP \to TP\}_{i=1}^r$ of constant rank satisfying properties* (2.4.1) *and* (2.4.2)(1)*.*

Proof. It is immediate to realize (2.4.3) imply (2.4.1) and (2.4.2)(1). The only thing that we have to prove is the G-equivariance of the χ^i's, but this follows from G-equivariance of the Σ^i_u's. □

Proposition 2.4.4. *Given an r-split structure on a principal bundle $P(M,G)$, every G-invariant vector field Ξ on P splits into r G-invariant vector fields $\left\{\underset{i}{\Xi}\right\}$ such that $\Xi = \bigoplus_{i=1}^r \underset{i}{\Xi}$ and $\underset{i}{\Xi}(u) \in \Sigma^i_u$ for all $u \in P$ and $i \in \{1,\dots,r\}$.*

Remark 2.4.5. The vector fields $\left\{\underset{i}{\Xi}\right\}$ are compatible with the $\{\Sigma^i\}$, i.e. they are sections $\left\{\underset{i}{\Xi}\colon P \to \Sigma^i\right\}$ of the vector bundles $\{\Sigma^i \to P\}$.

Proof of Proposition 2.4.4. If we set $\underset{i}{\Xi}(u) = \chi^i_u \circ \Xi(u)$ for all $u \in P$ and $i \in \{1,\dots,r\}$, then it follows immediately from (2.4.2)(1) and (2.4.3)(1) that Ξ splits as $\bigoplus_{i=1}^r \underset{i}{\Xi}$. Again, the only thing we have to check is the G-invariance of the $\underset{i}{\Xi}$'s. But now, from the G-equivariance of the χ^i's and the G-invariance of Ξ,

$$\begin{aligned} T_u\tilde{R}_a \circ \underset{i}{\Xi}(u) &= T_u\tilde{R}_a \circ \chi^i_u \circ \Xi(u) \\ &= T_u\tilde{R}_a \circ \chi^i_u \circ T_{u\cdot a}\tilde{R}_{a^{-1}} \circ T_u\tilde{R}_a \circ \Xi(u) \\ &= \chi^i_u \circ \Xi(u\cdot a) = \underset{i}{\Xi}(u\cdot a) \end{aligned}$$

for all $u \in P$, $a \in G$ and $i \in \{1,\dots,r\}$, which is the thesis. □

Example 2.4.6. Comparing with §1.5, we see that, for the split structure induced by a principal connection on a principal bundle $P(M,G)$, we have $r = 2$, $\chi^1 = \chi$, $\chi^2 = \varkappa$, $\Sigma^1 = HP$, $\Sigma^2 = VP$, $\underset{1}{\Xi} = \hat{\Xi}$ and $\underset{2}{\Xi} = \check{\Xi}$.

Corollary 2.4.7. *Let $P(M,G)$ be a reductive G-structure on a principal bundle $Q(M,H)$ and let $i_P \colon P \to Q$ be the canonical embedding. Then, any given r-split structure on $Q(M,H)$ induces an r-split structure restricted to $P(M,G)$, i.e. an equivariant decomposition of $i_P^*(TQ) \equiv P \times_Q TQ = \{\,(u,v) \in P \times TQ \mid \tau_Q(v) = i_P(u)\,\}$ such that $i_P^*(TQ) = \bigoplus_{i=1}^r i_P^*(\Sigma^i)$, and any H-invariant vector field Ξ on Q restricted to P splits* *into r G-invariant sections of the pull-back bundles* $\{i_P^*(\Sigma^i) \equiv P \times_Q \Sigma^i\}$, *i.e.* $\Xi|_P = \bigoplus_{i=1}^r \underset{i}{\Xi}$ *with* $\underset{i}{\Xi}(u) \in \left(i_P^*(\Sigma^i)\right)_u$ *for all* $u \in P$ *and* $i \in \{1, \dots, r\}$.

Proof. The first part of the corollary is obvious. The second part follows immediately from Proposition 2.4.4 once one realizes that the restriction to P of any (H-invariant) vector field Ξ on Q is a section of $i_P^*(TQ) \to P$. □

Remark 2.4.8. Note that the pull-back i_P^* is a *natural operation*, i.e. it respects the splitting $i_P^*(TQ) = \bigoplus_{i=1}^r i_P^*(\Sigma^i)$. In other words, the pull-back of a splitting for Q is a splitting of the pull-backs for P. Furthermore, although the vector fields $\left\{\underset{i}{\Xi} \colon P \to i_P^*(\Sigma^i)\right\}$ are G-invariant sections of their respective pull-back bundles, by virtue of Proposition 2.4.4 they are H-invariant if regarded as vector fields on the corresponding subsets of Q, i.e. as sections $\left\{\underset{i}{\Xi}_Q \colon Q \to \Sigma^i\right\}$ such that $\underset{i}{\Xi}_Q \circ i_P = Ti_P \circ \underset{i}{\Xi}$. Equivalently, $\underset{i}{\Xi} = \underset{i}{\Xi}_Q\Big|_P$. In the sequel, we shall not formally distinguish between $\underset{i}{\Xi}$ and $\underset{i}{\Xi}_Q$.

Now, in order to proceed any further, we first need a different characterization of principal bundles.

Definition 2.4.9 (Kobayashi & Nomizu 1963). Let M be a manifold and G a Lie group. A ***principal bundle*** over M with structure group G consists of a manifold P and an action of G on P satisfying the following conditions:

(i) G acts freely on P on the right: $(u, a) \in P \times G \mapsto u \cdot a \in P$;

(ii) M is the quotient space of P by the equivalence relation induced by G, i.e. M coincides with the space of orbits P/G (*cf.* §C.3), and the canonical projection $\pi \colon P \to M$ is smooth;

(iii) P is locally trivial, i.e. every point x of M has a neighbourhood U_α such that there is a diffeomorphism $\psi_\alpha \colon \pi^{-1}(U_\alpha) \to U_\alpha \times G$ such that $\psi_\alpha(u) = (\pi(u), f_\alpha(u))$ for all $u \in \pi^{-1}(U_\alpha)$, $f_\alpha \colon \pi^{-1}(U_\alpha) \to G$ being a mapping satisfying $f_\alpha(u \cdot a) = f_\alpha(u)a$ for all $u \in \pi^{-1}(U_\alpha)$, $a \in G$.

It is easy to see that this definition of a principal bundle is completely equivalent to the one given in §1.2: condition (i) defines the canonical right action on P introduced in §1.3, condition (ii) amounts to saying that (P, M, π) is a fibred manifold, whereas condition (iii) can be used to define the transition functions $a_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$ with values in G by $a_{\alpha\beta}(x) = f_\alpha(\pi^{-1}(x)) f_\beta(\pi^{-1}(x))^{-1}$ for all $x \in U_\alpha \cap U_\beta$. The reader is referred to Kobayashi & Nomizu (1963), Chapter I, §5, for more detail.

Now, in §1.10 we saw that $W^{k,h}P$ is a principal bundle over M. Consider in particular $W^{1,1}P$, the $(1,1)$-principal prolongation of P. The fibred manifold $W^{1,1}P \to M$ coincides with the fibred product $L^1M \times_M J^1P$ over M. We have two canonical principal bundle morphisms $\mathrm{pr}_1 \colon W^{1,1}P \to L^1M$ and $\mathrm{pr}_2 \colon W^{1,1}P \to P$ (*cf.* Kolář *et al.* 1993). In particular, $\mathrm{pr}_2 \colon W^{1,1}P \to P$ is a $G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m$-principal bundle, $G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m$ being the kernel of $W_m^{1,1}G \to G$. Indeed, recall from group theory that, if $f \colon G \to G'$ is a group epimorphism, then $G' \cong G/\ker f$. Therefore, if $\pi_{0,0}^{1,1}$ denotes the canonical projection from $W_m^{1,1}G$ to G, then $G \cong W^{1,1}_m G/\ker \pi^{1,1}_{0,0}$. Hence, recalling the previous definition of a principal bundle, we have:

$$W^{1,1}P/W^{1,1}_m G = M = P/G = P/(W^{1,1}_m G/\ker \pi^{1,1}_{0,0}),$$

from which we deduce that $\mathrm{pr}_2\colon W^{1,1}P \to P$ is a $(\ker \pi^{1,1}_{0,0})$-principal bundle. It remains to show that $\ker \pi^{1,1}_{0,0} \cong G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m$, but this is obvious if we consider that $\ker \pi^{1,1}_{0,0}$ is coordinatized by $(\alpha^j{}_k, e^{\mathcal{B}}, a^{\mathcal{C}}{}_l)$ (*cf.*, e.g., Example 1.10.16). □

The following lemma recognizes $\tau_P\colon TP \to P$ as a vector bundle associated with the principal bundle $W^{1,1}P \to P$.

Lemma 2.4.10. *The vector bundle $\tau_P\colon TP \to P$ is isomorphic to the vector bundle $T^{1,1}P := (W^{1,1}P \times \mathcal{V})/(G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m)$ over P, where $\mathcal{V} := \mathbb{R}^m \oplus \mathfrak{g}$ is the left $G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m$-manifold with action given by:*

$$\begin{cases} \lambda\colon (G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m) \times \mathcal{V} \to \mathcal{V} \\ \lambda\colon \Big((\alpha^j{}_k, e^{\mathcal{B}}, a^{\mathcal{C}}{}_l), (v^i, w^{\mathcal{A}})\Big) \mapsto (\alpha^i{}_j v^j, w^{\mathcal{A}} + a^{\mathcal{A}}{}_i v^i) \end{cases}. \tag{2.4.4}$$

Proof. It is easy to show that the tangent bundle TG of a Lie group G is again a Lie group, and, if $P(M,G)$ is a principal bundle, so is $TP(TM,TG)$ (*cf.*, e.g, Kolář *et al.* 1993, §10). Now, the canonical right action $\tilde{R}$ on P induces a canonical right action on TP simply given by $T\tilde{R}$. It is then easy to realize that the space of orbits TP/G, regarded as vector bundle over M, is canonically isomorphic to the bundle of G-invariant vector fields on P (*cf.* §1.3). Hence, taking Example 1.10.18 into account, we have:

$$TP/(W^{1,1}_m G/G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m) \cong TP/G \cong (W^{1,1}P \times \mathcal{V})/W^{1,1}_m G,$$

from which it follows that $\tau_P\colon TP \to P$ is a [gauge-natural vector] bundle [of order $(0,0)$] associated with $\mathrm{pr}_2\colon W^{1,1}P \to P$. Action (2.4.4) is nothing but action (1.10.9) restricted to $G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m$. □

Lemma 2.4.11. *$VP \to P$ is a trivial vector bundle associated with $W^{1,1}P \to P$.*

Proof. We already know that $VP \to P$ is a trivial vector bundle (*cf.* §1.5). To see that it is associated with $W^{1,1}P \to P$, we follow the same argument as before, this time taking into account Example 1.10.19. We then have:

$$VP/(W^{1,1}_m G/G^1_m \rtimes \mathfrak{g} \otimes \mathbb{R}^m) \cong VP/G \cong (W^{1,1}P \times \mathfrak{g})/W^{1,1}_m G,$$

whence the result follows. □

Lemma 2.4.12. *Let $P(M,G)$ be a reductive G-structure on a principal bundle $Q(M,H)$ and $i_P\colon P \to Q$ the canonical embedding. Then, $i^*_P(TQ) = P \times_Q TQ$ is a vector bundle over P associated with $W^{1,1}P \to P$.*

Proof. It follows immediately from Lemma 2.4.10 once one realizes that $i^*_P(TQ)$ is by definition a vector bundle over P with fibre $\mathbb{R}^m \oplus \mathfrak{h}$ and the same structure group as $TP \to P$ (see also Figure 2.4.1 below). □

From the above lemmas it follows that another important example of a split structure on a principal bundle is given by the following

Theorem 2.4.13. *Let $P(M,G)$ be a reductive G-structure on a principal bundle $Q(M,H)$ and let $i_P\colon P \to Q$ be the canonical embedding. Then, there exists a canonical decomposition of $i_P^*(TQ) \to P$ such that*

$$i_P^*(TQ) \cong TP \oplus_P \mathcal{M}(P),$$

i.e. at each $u \in P$ one has

$$T_uQ \cong T_uP \oplus \mathcal{M}_u,$$

$\mathcal{M}_u$ being the fibre over u of the subbundle $\mathcal{M}(P) \to P$ of $i_P^*(VQ) \to P$. The bundle $\mathcal{M}(P)$ is defined as $\mathcal{M}(P) := (W^{1,1}P \times \mathfrak{m})/(G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m)$, where $\mathfrak{m}$ is the (trivial left) $G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m$-manifold.

Proof. From Lemma 2.4.12 and the fact that G is a reductive Lie subgroup of H (Definition 2.3.1) it follows that

$$\begin{aligned} i_P^*(TQ) &\cong \big(W^{1,1}P \times (\mathbb{R}^m \oplus \mathfrak{h})\big)/(G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m) \\ &= \big(W^{1,1}P \times (\mathbb{R}^m \oplus \mathfrak{g})\big)/(G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m) \oplus_P (W^{1,1}P \times \mathfrak{m})/(G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m) \\ &\equiv TP \oplus_P \mathcal{M}(P). \end{aligned}$$

The trivial $G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m$-manifold $\mathfrak{m}$ corresponds to action (1.10.10) of Example 1.10.19 with $W_m^{1,1}G$ restricted to $G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m$, and $\mathfrak{g}$ restricted to $\mathfrak{m}$. Of course, since the group $G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m$ acts trivially on $\mathfrak{m}$, it follows that $\mathcal{M}(P)$ is trivial, i.e. isomorphic to $P \times \mathfrak{m}$, because $W^{1,1}P/(G_m^1 \rtimes \mathfrak{g} \otimes \mathbb{R}^m) \cong P$. □

From the above theorem two corollaries follow, which are of prime importance for the concepts of a Lie derivative we shall introduce in the next section.

Corollary 2.4.14. *Let $P(M,G)$ and $Q(M,H)$ be as in the previous theorem. The restriction $\Xi|_P$ of an H-invariant vector field Ξ on Q to P splits into a G-invariant vector field Ξ_{K} on P, called the* **Kosmann vector field associated with** Ξ, *and a "transverse" vector field Ξ_{G}, called the* **von Göden vector field associated with** Ξ.

The situation is schematically depicted in Figure 2.4.1 [Q is represented as a straight line, and P as the half-line stretching to the mark; TQ is represented as a parallelepiped over Q, $i_P^*(TQ)$ as the part of it corresponding to P, whereas TP is the face of $i_P^*(TQ)$ facing the reader].

Corollary 2.4.15. *Let $P(M,G)$ be a classical G-structure, i.e. a reductive G-structure on the bundle LM of linear frames over M. The restriction $L\xi|_P$ to $P \to M$ of the natural lift $L\xi$ onto LM of a vector field ξ on M splits into a G-invariant vector field on P called the* **generalized Kosmann lift of** ξ *and denoted simply by ξ_{K}, and a "transverse" vector field called the* **von Göden lift of** ξ *and denoted by ξ_{G}.*

Remark 2.4.16. The last corollary still holds if, instead of LM, one considers the k-th order frame bundle L^kM and hence a classical G-*structure of order* k, i.e. a reductive G-subbundle P of L^kM.

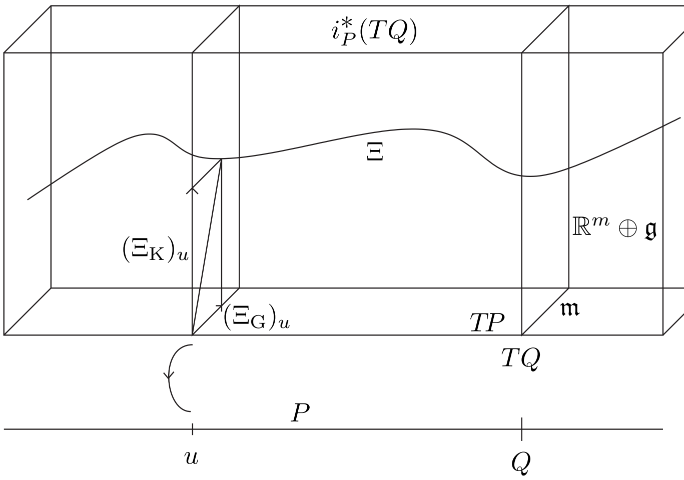


Figure 2.4.1: Kosmann and von Göden vector fields.

Example 2.4.17 (Kosmann lift). A fundamental example of a G-structure on a manifold M is given, of course, by the bundle $\mathrm{SO}(M,g)$ of its (pseudo-) orthonormal frames with respect to a metric g of signature (p,q), where $p+q=m\equiv\dim M$. $\mathrm{SO}(M,g)$ is a principal bundle (over M) with structure group $G=\mathrm{SO}(p,q)^e$. Now, combining (1.6.3) and (1.10.5), we can define the natural lift $L\xi$ of a vector field ξ on M onto LM as

$$L\xi := L^1\xi \equiv \frac{\partial}{\partial t}L^1\varphi_t\bigg|_{t=0}, \tag{2.4.5}$$

$\{\varphi_t\}$ denoting the flow of ξ. If $(\rho_a{}^b)$ denotes a (local) basis of right $\mathrm{GL}(m,\mathbb{R})$-invariant vector fields on LM reading $(\rho_a{}^b = u^b{}_c\partial/\partial u^a{}_c)$ in some local chart $(x^\mu, u^a{}_b)$ and $(e_a =: e_a{}^\mu\partial_\mu)$ is a local section of LM (*cf.* §1.3.1), then $L\xi$ has the local expression

$$L\xi = \xi^a e_a + (L\xi)^a{}_b\rho_a{}^b,$$

where $\xi =: \xi^a e_a =: \xi^\mu\partial_\mu$ and

$$(L\xi)^a{}_b := \theta^a{}_\rho(\partial_\nu\xi^\rho e_b{}^\nu - \xi^\nu\partial_\nu e_b{}^\rho). \tag{2.4.6}$$

Indeed, on locally expanding φ_t as $\varphi_t^\mu(x^\lambda) = x^\mu + t\xi^\mu(x^\lambda) + O^\mu(t^2)$ and applying (2.4.5), we immediately find $(L\xi)^\mu{}_\nu = \partial_\nu\xi^\mu$. Hence, on using (1.3.14), we recover precisely (2.4.6). If we now let (e_a) and $(x^\mu, u^a{}_b)$ denote a local section and a local chart of $\mathrm{SO}(M,g)$, respectively, then the generalized Kosmann lift ξ_{K} on $\mathrm{SO}(M,g)$ of a vector field ξ on M, simply called its ***Kosmann lift*** (Fatibene *et al.* 1996), locally reads

$$\xi_{\mathrm{K}} = \xi^a e_a + (L\xi)_{[ab]}A^{ab}, \tag{2.4.7}$$

where (A^{ab}) is a basis of right $\mathrm{SO}(p,q)^e$-invariant vector fields on $\mathrm{SO}(M,g)$ locally reading $(A^{ab} = \eta^{c[a}\delta^{b]}{}_d\rho_c{}^d)$, $(L\xi)_{ab} := \eta_{ac}(L\xi)^c{}_b$, and (η_{ac}) denote the components of the standard Minkowski metric of signature (p,q) (*cf.* §C.1).

Now, combining Proposition 2.3.9 and Theorem 2.4.13 yields the following result, which, in particular, will enable us to extend the concept of a Kosmann lift to the important context of spinor fields.

Corollary 2.4.18. *Let* $\zeta\colon \tilde{P} \to P$ *be a* Γ*-structure over a classical* G*-structure* $P(M,G)$*. Then, the generalized Kosmann lift* ξ_{K} *of a vector field* ξ *on* M *lifts to a unique* (Γ*-invariant*) *vector field* $\tilde{\xi}_{\mathrm{K}}$ *on* $\tilde{P}$*, which projects on* ξ_{K}*.*

2.5 Lie derivatives on reductive G-structures: the Lie derivative of spinor fields

Remark 2.5.1 (Notation). Let $P(M,G)$ be a principal bundle and recall Definition 2.1.4. For each Γ-structure $\zeta\colon \tilde{P} \to P$ on P, we shall simply write $\pounds_{\Xi}\tilde{\sigma} := \pounds_{\tilde{\Xi}}\tilde{\sigma}\colon M \to \bar{\tilde{P}}_{\tilde{\lambda}}$ for the corresponding (restricted) gauge-natural Lie derivative, $\tilde{P}_{\tilde{\lambda}}$ denoting a gauge-natural bundle associated with $\tilde{P}$ admitting a canonical vertical splitting $V\tilde{P}_{\tilde{\lambda}} \cong \tilde{P}_{\tilde{\lambda}} \times_M \bar{\tilde{P}}_{\tilde{\lambda}}$ and $\tilde{\sigma}\colon M \to \tilde{P}_{\tilde{\lambda}}$ one of its sections, which makes sense since Ξ admits a unique (Γ-invariant) lift $\tilde{\Xi}$ onto $\tilde{P}$ (Proposition 2.3.9).

Of course, we can now further specialize to the case of classical G-structures and, in particular, give the following

Definition 2.5.2. Let P_λ be a gauge-natural bundle associated with some classical G-structure $P(M,G)$, ξ_{K} the generalized Kosmann lift (on P) of a vector field ξ on M, and $\sigma\colon M \to P_\lambda$ a section of P_λ. Then, by the ***generalized Lie derivative of*** σ ***with respect to*** ξ_{K} we shall mean the generalized gauge-natural Lie derivative $\tilde{\pounds}_{\xi_{\mathrm{K}}}\sigma$ of σ with respect to ξ_{K} in the sense of Definition 2.1.4.

Consistently with (2.1.5) and Remark 2.5.1, we shall simply write $\pounds_{\xi_{\mathrm{K}}}\sigma\colon M \to \bar{P}_\lambda$ for the corresponding restricted Lie derivative, whenever defined, and $\pounds_{\xi_{\mathrm{K}}}\tilde{\sigma} := \pounds_{\tilde{\xi}_{\mathrm{K}}}\tilde{\sigma}\colon M \to \bar{\tilde{P}}_{\tilde{\lambda}}$ for the (restricted) Lie derivative of a section σ of a gauge-natural bundle $\tilde{P}_{\tilde{\lambda}}$ associated with some Γ-structure $\zeta\colon \tilde{P} \to P$ (and admitting a canonical vertical splitting), which makes sense since ξ_{K} admits a unique (Γ-invariant) lift $\tilde{\xi}_{\mathrm{K}}$ onto $\tilde{P}$ (Corollary 2.4.18).

Example 2.5.3 (Lie derivative of spinor fields. I). In Example 2.4.17 we mentioned that a fundamental example of a G-structure on a manifold M is given by the bundle $\mathrm{SO}(M,g)$ of its (pseudo-) orthonormal frames. An equally fundamental example of a Γ-structure on $\mathrm{SO}(M,g)$ is given by the corresponding spin bundle $\mathrm{Spin}(M,g)$ with structure group $\Gamma = \mathrm{Spin}(p,q)^e$ (*cf.* §D.2). Now, it is obvious that spinor fields can be regarded as sections of a suitable gauge-natural bundle over M. Indeed, if $\hat{\gamma}$ is the linear representation of $\mathrm{Spin}(p,q)^e$ on the vector space $\mathbb{C}^m$ induced by a given choice of γ matrices, then the associated vector bundle $\mathrm{Spin}(M,g)_{\hat{\gamma}} := \mathrm{Spin}(M,g) \times_{\hat{\gamma}} \mathbb{C}^m$ is a gauge-natural bundle of order $(0,0)$ whose sections represent spinor fields (*cf.* §D.2). Therefore, in spite of what is sometimes believed, a Lie derivative of spinors (in the sense of Definition 2.1.4) always exists, *no matter what* the vector field ξ on M is. By virtue of (2.1.10) and (D.1.6), such a Lie derivative locally reads

$$\pounds_{\Xi}\psi = \xi^a e_a\psi + \frac{1}{4}\Xi_{ab}\gamma^a\gamma^b\psi \tag{2.5.1}$$

for any spinor field ψ, ($\Xi_{ab} = \Xi_{[ab]}$) denoting the components of an $\mathrm{SO}(p,q)^e$-invariant vector field $\Xi = \xi^a e_a + \Xi_{ab} A^{ab}$ on $\mathrm{SO}(M,g)$, $\xi =: \xi^a e_a$, and $e_a\psi$ the "Pfaff derivative" of ψ along the local section ($e_a =: e_a{}^\mu \partial_\mu$) of $\mathrm{SO}(M,g)$ induced by some local section of $\mathrm{Spin}(M,g)$. This is the most general notion of a (gauge-natural) Lie derivative of spinor fields and the appropriate one for most situations of physical interest (*cf.* Godina *et al.* 2001; Matteucci 2002): the generality of Ξ might be disturbing, but is the *unavoidable* indication that $\mathrm{Spin}(M,g)_{\hat\gamma}$ is *not* a natural bundle.

If we wish nonetheless to remove such a generality, we must *choose* some canonical (*not* natural) lift of ξ onto $\mathrm{SO}(M,g)$. The conceptually (*not* mathematically) most "natural" choice is perhaps given by the Kosmann lift (recall Example 2.4.17 and use Corollary 2.4.18). The ensuing Lie derivative locally reads

$$\pounds_{\xi_{\mathrm{K}}}\psi = \xi^a e_a\psi + \frac{1}{4}(L\xi)_{[ab]}\gamma^a\gamma^b\psi. \tag{2.5.2}$$

Of course, if '∇' and '$\tilde\nabla$' denote the covariant derivative operators associated with the connection on $\mathrm{Spin}(M,g)$ and the transposed connection on TM, respectively, induced by a given principal connection on $\mathrm{SO}(M,g)$, it easy to see that, on using (D.2.4) and (1.5.18), the previous expression can be recast into the form

$$\pounds_{\xi_{\mathrm{K}}}\psi = \xi^a \nabla_a\psi - \frac{1}{4}\tilde\nabla_{[a}\xi_{b]}\gamma^a\gamma^b\psi, \tag{2.5.2$'$}$$

which, in the case of a symmetric connection where $\tilde\nabla = \nabla$, reproduces exactly Kosmann's (1972) definition.

Remark 2.5.4. An easy calculation shows that the Kosmann lift [onto $\mathrm{SO}(M,g)$] satisfies the "quasi-naturality" condition (*cf.* Bourguignon & Gauduchon 1992; Fatibene 1999)

$$[\xi_{\mathrm{K}}, \eta_{\mathrm{K}}] = [\xi,\eta]_{\mathrm{K}} + \mathrm{K}(\xi,\eta), \tag{2.5.3}$$

where $K(\xi,\eta)$ is a vertical vector field on $\mathrm{SO}(M,g)$ locally given by

$$\mathrm{K}(\xi,\eta) \equiv \frac{1}{4}[(\pounds_\xi g)^\#, (\pounds_\eta g)^\#]^a{}_b A_a{}^b,$$

$(\pounds_\xi g)^\#$ denoting the endomorphism corresponding to $\pounds_\xi g$. Then, if we set $\pounds^{\mathrm{K}}_\xi := \pounds_{\xi_{\mathrm{K}}}$, we have

$$[\pounds^{\mathrm{K}}_\xi, \pounds^{\mathrm{K}}_\eta] = [\pounds_{\xi_{\mathrm{K}}}, \pounds_{\eta_{\mathrm{K}}}] = \pounds_{[\xi_{\mathrm{K}},\eta_{\mathrm{K}}]} = \pounds_{[\xi,\eta]_{\mathrm{K}}+K(\xi,\eta)} = \pounds^{\mathrm{K}}_{[\xi,\eta]} - \mathrm{K}_\lambda(\xi,\eta), \tag{2.5.4}$$

where the second identity follows from (2.2.5), the third one from (2.5.3), and $\mathrm{K}_\lambda(\xi,\eta)$ is the gauge-natural lift of $\mathrm{K}(\xi,\eta)$ onto the given *gauge*-natural vector (or affine) bundle $\mathrm{SO}(M,g)_\lambda \to M$ [$\mathrm{Spin}(M,g)_\lambda \to M$] associated with $\mathrm{SO}(M,g)$ [$\mathrm{Spin}(M,g)$]. A glance at (2.5.4) reveals that $\pounds^{\mathrm{K}}$ will be a Lie algebra homomorphism from $\mathfrak{X}(M)$ to $\mathrm{End}C^\infty(\mathrm{SO}(M,g)_\lambda)$ [$\mathrm{End}C^\infty(\mathrm{Spin}(M,g)_\lambda)$] for a gauge-natural vector (or affine) bundle $\mathrm{SO}(M,g)_\lambda$ [$\mathrm{Spin}(M,g)_\lambda$] *only if* ξ and/or η are conformal Killing vector fields (*cf.* §B.1). Thus, $\pounds^{\mathrm{K}}$ can be regarded as a Lie algebra homomorphism from the Lie algebra of conformal Killing vector fields on M to $\mathrm{End}C^\infty(\mathrm{SO}(M,g)_\lambda)$ [$\mathrm{End}C^\infty(\mathrm{Spin}(M,g)_\lambda)$]. Note that, on a (purely) *natural* bundle, $[\pounds^{\mathrm{K}}_{\xi}, \pounds^{\mathrm{K}}_{\eta}] = \pounds^{\mathrm{K}}_{[\xi,\eta]}$, since $\pounds_{\xi_{\mathrm{K}}}$ must then reduce to the ordinary (natural) Lie derivative $\pounds_{\xi}$, for which (2.2.6) holds.

Remark 2.5.5. We stress that, although *in this case* its local expression would be identical with (2.5.2), $\pounds_{\xi_{\mathrm{K}}}$ is *not* the "metric Lie derivative" introduced by Bourguignon & Gauduchon (1992). To convince oneself of this it is enough to take the Lie derivative of the metric g, which is a section of the *natural* bundle $\bigvee^2 T^*M$, '$\bigvee$' denoting the symmetrized tensor product. Since the (restricted) Lie derivative $\pounds_{\xi_{\mathrm{K}}}$ in the sense of Definition 2.5.2 must reduce to the ordinary one on natural objects, it holds that

$$\pounds_{\xi} g = \pounds_{\xi_{\mathrm{K}}} g.$$

On the other hand, if $\pounds_{\xi_{\mathrm{K}}}$ coincided with the operator $\pounds^{g}_{\xi}$ defined by Bourguignon & Gauduchon (1992), the right-hand side of the above identity should equal zero (Bourguignon & Gauduchon 1992, Proposition 15), thereby implying that ξ is a Killing vector field, contrary to the fact that ξ is completely arbitrary. Indeed, in order to recover Bourguignon and Gauduchon's definition, another concept of a Lie derivative must be introduced.

We shall start by recalling two classical definitions due to Kobayashi (1972).

Definition 2.5.6. Let $P(M, G)$ be a (classical) G-structure. Let φ be a diffeomorphism of M onto itself and $L^1\varphi$ its natural lift onto LM. If $L^1\varphi$ maps P onto itself, i.e. if $L^1\varphi(P) \subseteq P$, then φ is called an ***automorphism of the G-structure*** P.

Definition 2.5.7. Let $P(M, G)$ be a G-structure. A vector field ξ on M is called an ***infinitesimal automorphism of the G-structure*** P if it generates a local one-parameter group of automorphisms of P.

We can now generalize these concepts to the framework of reductive G-structures as follows.

Definition 2.5.8. Let P be a reductive G-structure on a principal bundle $Q(M, H)$ and Φ a principal automorphism of Q. If Φ maps P onto itself, i.e. if $\Phi(P) \subseteq P$, then Φ is called a ***generalized automorphism of the reductive G-structure*** P.

Clearly, since Φ is H-equivariant by definition, its restriction $\Phi|_P \colon P \to P$ to P is G-equivariant, and hence is a principal automorphism of P. Then, each element of $\mathrm{Aut}(P)$, i.e. each principal automorphism of P, is automatically a generalized automorphism of the reductive G-structure P. Analogously, we have

Definition 2.5.9. Let P be a reductive G-structure on a principal bundle $Q(M, H)$. An H-invariant vector field Ξ on Q is called a ***generalized infinitesimal automorphism of the reductive G-structure*** P if it generates a local one-parameter group of generalized automorphisms of P.

Clearly, the restriction $\Xi|_P$ of Ξ to P is a G-invariant vector field on P, and, on the other hand, each element of $\mathfrak{X}_G(P)$, i.e. each G-invariant vector field on P, is, trivially, a generalized infinitesimal automorphism of the reductive G-structure P. Kobayashi's classical definitions are then recovered on setting $Q = LM$, $H = \mathrm{GL}(m, \mathbb{R})$, $\Phi = L^1\varphi$ and $\Xi = L\xi$.

Proposition 2.5.10. *Let $P(M, G)$ be a reductive G-structure on a principal bundle $Q(M, H)$. An H-invariant vector field Ξ on Q is a generalized infinitesimal automorphism of the reductive G-structure P if and only if Ξ is tangent to P at each point of P.*

Proof. Let $\{\Phi_t\}$ be the flow of Ξ. Then, from Definition 2.5.9 it follows that Φ_t is a generalized automorphism of P for all t. Hence, Φ_t maps P into itself, and this is clearly equivalent to Ξ being tangent to P at each point of P. This result generalizes Proposition 1.1 of Kobayashi & Nomizu (1969), Chapter X, which is easily recovered on setting $\Xi = L\xi$. □

We then have the following important

Lemma 2.5.11. *Let P be a reductive G-structure on a principal bundle $Q(M, H)$ and Ξ a generalized infinitesimal automorphism of the reductive G-structure P. Then, the flow $\{\Phi_t\}$ of Ξ, it being H-invariant, induces on each gauge-natural bundle Q_λ associated with Q a one-parameter group $\{(\Phi_t)_\lambda\}$ of global automorphisms.*

Proof. Since Ξ is by assumption a generalized infinitesimal automorphism, it is by definition an H-invariant vector field on Q. Therefore, its flow $\{\Phi_t\}$ is a one-parameter group of H-equivariant maps on Q. Then, if $Q_\lambda = W^{k,h}Q \times_\lambda F$, we set

$$(\Phi_t)_\lambda([u, f]_\lambda) := [W^{k,h}\Phi_t(u), f]_\lambda,$$

$(u, f) \in Q \times F$, and are back to the situation of formula (1.10.4). □

Corollary 2.5.12. *Let P and $Q(M, H)$ be as in the previous lemma, and let Ξ be an H-invariant vector field on Q. Then, the flow $\{(\Phi_{\mathrm{K}})_t\}$ of the generalized Kosmann vector field Ξ_{K} associated with Ξ induces on each gauge-natural bundle Q_λ associated with Q a one-parameter group $\left\{\left((\Phi_{\mathrm{K}})_t\right)_\lambda\right\}$ of global automorphisms.*

Proof. Recall that, although the generalized Kosmann vector field Ξ_{K} is a G-invariant vector field on P, it is H-invariant if regarded as a vector field on the corresponding subset of Q (*cf.* Remark 2.4.8 and Corollary 2.4.14). Therefore, its flow $\{(\Phi_{\mathrm{K}})_t\}$ is a one-parameter group of H-equivariant automorphisms on the subset P of Q.
We now want to define a one-parameter group of automorphisms $\left\{\left((\Phi_{\mathrm{K}})_t\right)_\lambda\right\}$ of $Q_\lambda = W^{k,h}Q \times_\lambda F$. Let $[u, f]_\lambda \in Q_\lambda$, $u \in Q$ and $f \in F$, and let u_1 be a point in P such that $\pi(u_1) = \pi(u)$, $\pi\colon Q \to M$ denoting the canonical projection. There exists a unique $a_1 \in H$ such that $u = u_1 \cdot a_1$. Set then

$$\left((\Phi_{\mathrm{K}})_t\right)_\lambda([u, f]_\lambda) := [W^{k,h}(\Phi_{\mathrm{K}})_t(u_1), a_1 \cdot f]_\lambda.$$

We must show that, given another point $u_2 \in P$ such that $u = u_2 \cdot a_2$ for some (unique) $a_2 \in H$, we have

$$[W^{k,h}(\Phi_{\mathrm{K}})_t(u_1), a_1 f]_\lambda = [W^{k,h}(\Phi_{\mathrm{K}})_t(u_2), a_2 \cdot f]_\lambda.$$

Indeed, since the action of H is free and transitive on the fibres, from $u = u_1 \cdot a_1$ and

$u = u_2 \cdot a_2$ it follows that $a_1 = aa_2$ or $a = a_1 a_2^{-1}$ or $a_2 = a^{-1}a_1$. But then

$$\begin{aligned}[W^{k,h}(\Phi_{\mathrm{K}})_t(u_2), a_2 f]_\lambda &= [W^{k,h}(\Phi_{\mathrm{K}})_t(u_1 \cdot a), (a^{-1}a_1)\cdot f]_\lambda \\ &= [W^{k,h}(\Phi_{\mathrm{K}})_t(u_1) \odot W^{k,h}_m a, a^{-1}(a_1 \cdot f)]_\lambda \\ &= [W^{k,h}(\Phi_{\mathrm{K}})_t(u_1), a_1 \cdot f]_\lambda,\end{aligned}$$

as claimed. It is then easy to see that the so-defined $\big((\Phi_{\mathrm{K}})_t\big)_\lambda$ does not depend on the chosen representative. □

By virtue of the previous corollary, we can now give the following

Definition 2.5.13. Let P be a reductive G-structure on a principal bundle $Q(M,H)$, $G \neq \{e\}$, and Ξ an H-invariant vector field on Q projecting over a vector field ξ on M. Let Q_λ be a gauge-natural bundle associated with Q and $\sigma\colon M \to Q_\lambda$ a section of Q_λ. Then, by the ***generalized G-reductive Lie derivative of σ with respect to*** Ξ we shall mean the map

$$\tilde{\pounds}^G_\Xi \sigma := \frac{\partial}{\partial t}\Big(\big((\Phi_{\mathrm{K}})_{-t}\big)_\lambda \circ \sigma \circ \varphi_t\Big)\Big|_{t=0},$$

$\{\varphi_t\}$ denoting the flow of ξ.

The corresponding notions of a restricted Lie derivative and a (generalized or restricted) Lie derivative on an associated Γ-structure (which will still be called "G-reductive" and denoted by the superscript G) can be defined in the usual way.

Remark 2.5.14. Of course, since Ξ_{K} is by definition a G-invariant vector field on P, Definition 2.5.13 makes sense also when σ is a section of a gauge-natural bundle P_λ associated with P, for which one does not even need Corollary 2.5.12.

Remark 2.5.15. When $Q = P$ (and $H = G$), Ξ_{K} is just Ξ, and we recover the notion of a (generalized) Lie derivative in the sense of Definition 2.1.4, *but*, as G is required not to equal the trivial group $\{e\}$, Q_λ is never allowed to be a (purely) natural bundle.

By its very definition, the (restricted) G-reductive Lie derivative does *not* reduce, in general, to the ordinary (natural) Lie derivative on fibre bundles associated with L^kM. This fact makes it unsuitable in all those situations where one needs a *unique* operator which reproduce "standard results" when applied to "standard objects".

In other words, $\pounds^G_\Xi$ is defined with respect to some *pre-assigned* (generalized) symmetries. We shall make this statement explicit in Proposition 2.5.17 below, which provides a generalization of a well-known classical result.

Let then K be a tensor over the vector space $\mathbb{R}^m$ (i.e., an element of the tensor algebra over $\mathbb{R}^m$) and G the group of linear transformations of $\mathbb{R}^m$ leaving K invariant. Recall that each reduction of the structure group $\mathrm{GL}(m,\mathbb{R})$ to G gives rise to a tensor field K on M. Indeed, we may regard each $u \in LM$ as a linear isomorphism of $\mathbb{R}^m$ onto T_xM, where $x = \pi(u)$ and $\pi\colon LM \to M$ denotes, as usual, the canonical projection. Now, if $P(M,G)$ is a G-structure, at each point x of M we can choose a frame u belonging to P such that $\pi(u) = x$. Since u is a linear isomorphism of $\mathbb{R}^m$ onto the tangent space T_xM, it induces an isomorphism of the tensor algebra over $\mathbb{R}^m$ onto the tensor algebra over T_xM. Then K_x is the image of K under this isomorphism. The invariance of K by G implies that K_x is defined independent of the choice of u in $\pi^{-1}(x)$. Then, the following proposition is evident.

Proposition 2.5.16 (Kobayashi 1972). *Let K be a tensor over the vector space $\mathbb{R}^m$ and G the group of linear transformations of $\mathbb{R}^m$ leaving K invariant. Let P be a G-structure on M and K the tensor field on M defined by K and P. Then,*

(i) *a diffeomorphism $\varphi\colon M \to M$ is an automorphism of the G-structure P iff φ leaves K invariant*;

(ii) *a vector field ξ on M is an infinitesimal automorphism of P iff $\pounds_\xi K = 0$.*

An analogous result for generalized automorphisms of P follows.

Proposition 2.5.17. *With the same hypotheses as the previous proposition,*

(i) *an automorphism $\Phi\colon LM \to LM$ is a generalized automorphism of the G-structure P iff Φ leaves K invariant*;

(ii) *a* $\mathrm{GL}(m,\mathbb{R})$*-invariant vector field Ξ on LM is an infinitesimal generalized automorphism of P iff $\pounds_\Xi K = 0$*;

(iii) $\pounds^G_\Xi K \equiv 0$ *for any* $\mathrm{GL}(m,\mathbb{R})$*-invariant vector field Ξ on LM.*

Proof. First, note that, here, K is regarded as a section of a *gauge*-natural, not simply natural, bundle over M (*cf.* Example 1.10.15). Then, since K is G-invariant, an automorphism $\Phi\colon LM \to LM$ will leave K unchanged if and only if it maps P onto itself and is G-equivariant on P, i.e. iff it is a generalized automorphism of P, whence (i) follows. Part (ii) is just the infinitesimal version of (i), whereas (iii) follows from (ii) and Definition 2.5.13 since Ξ_{K} is by definition a G-invariant vector field on P and hence, in particular, a generalized automorphism of P. The choice $\Xi = L\xi$ reproduces Kobayashi's (1972) classical result, which can therefore be stated in a (purely) natural setting, as in Proposition 2.5.16. □

Corollary 2.5.18. *Let Ξ be a* $\mathrm{GL}(m,\mathbb{R})$*-invariant vector field on LM, and let g be a metric tensor on M of signature (p,q). Then, $\pounds^{\mathrm{SO}(p,q)^e}_\Xi g \equiv 0$.*

Proof. It follows immediately from Proposition 2.5.17(iii). At the end of §2.6 we shall also give an explicit proof of this result using local coordinates. □

Corollary 2.5.18 suggests that Bourguignon & Gauduchon's (1992) metric Lie derivative might be a particular instance of an $\mathrm{SO}(p,q)^e$-reductive Lie derivative. This is precisely the case, as explained in the following fundamental

Example 2.5.19 (Lie derivative of spinor fields. II). We know that the Kosmann lift ξ_{K} onto $\mathrm{SO}(M,g)$ of a vector field ξ on M is an $\mathrm{SO}(p,q)^e$-invariant vector field on $\mathrm{SO}(M,g)$, and hence its lift $\tilde{\xi}_{\mathrm{K}}$ onto $\mathrm{Spin}(M,g)$ is a $\mathrm{Spin}(p,q)^e$-invariant vector field. As the spinor bundle $\mathrm{Spin}(M,g)_{\hat{\gamma}}$ is a vector bundle associated with $\mathrm{Spin}(M,g)$, the $\mathrm{SO}(p,q)^e$-reductive Lie derivative $\pounds^{\mathrm{SO}(p,q)^e}_{L\xi}\psi$ of a spinor field ψ coincides with $\pounds_{\xi_{\mathrm{K}}}\psi$, i.e.

locally with expression (2.5.2) or (2.5.2′). Indeed, in this case we have, with an obvious notation, $Q = LM$, $H = \mathrm{GL}(m, \mathbb{R})$, $P = \mathrm{SO}(M, g)$, $G = \mathrm{SO}(M, g)$, $\tilde{P} = \mathrm{Spin}(M, g)$, $\Gamma = \mathrm{Spin}(p, q)^e$ and $\tilde{P}_{\tilde{\lambda}} = \mathrm{Spin}(M, g)_{\hat{\gamma}}$ (*cf.* Definition 2.5.13 and Remark 2.5.14).
Now recall that, in order to evaluate the $\mathrm{SO}(p, q)^e$-reductive Lie derivative $\pounds_{L\xi}^{\mathrm{SO}(p,q)^e} g$ of g, we must regard g as a *gauge*-natural, not simply natural, object. Then, if $g = g_{\mu\nu}\, \mathrm{d}x^\mu \vee \mathrm{d}x^\nu$ in some natural chart, we have the local expression

$$\begin{aligned}\pounds_{L\xi}^{\mathrm{SO}(p,q)^e} g_{\mu\nu} &\equiv \xi^\rho \partial_\rho g_{\mu\nu} + 2 g_{\rho(\mu} (\xi_{\mathrm{K}})^\rho{}_{\nu)} \\ &\equiv \xi^\rho \partial_\rho g_{\mu\nu} + g_{\rho(\mu} \partial_{\nu)} \xi^\rho - \delta^\rho{}_{(\mu} g_{\nu)\sigma} \partial_\rho \xi^\sigma - \xi^\rho \delta^\sigma{}_{(\mu|} \partial_\rho g_{|\nu)\sigma} \\ &\equiv 0,\end{aligned}$$

as required by Corollary 2.5.18, the $(\xi_{\mathrm{K}})^\rho{}_\nu$'s having been obtained from the $(\xi_{\mathrm{K}})^a{}_b \equiv \eta^{ac}(L\xi)_{[cb]}$'s on using (2.4.6) and (1.3.14′). This is, of course, quite different from the usual (*natural*) Lie derivative

$$\begin{aligned}\pounds_\xi g_{\mu\nu} &\equiv \xi^\rho \partial_\rho g_{\mu\nu} + 2 g_{\rho(\mu} (L\xi)^\rho{}_{\nu)} \\ &\equiv \xi^\rho \partial_\rho g_{\mu\nu} + 2 g_{\rho(\mu} \partial_{\nu)} \xi^\rho \\ &\equiv 2 \nabla_{(\mu} \xi_{\nu)} \\ &\equiv \pounds_{\xi_{\mathrm{K}}} g_{\mu\nu} \equiv \pounds_\Xi g_{\mu\nu},\end{aligned}$$

where the metric is regarded as a *purely* natural object. Hence, we can identify Bourguignon & Gauduchon's (1992) metric Lie derivative $\pounds_\xi^g$ with $\pounds_{L\xi}^{\mathrm{SO}(p,q)^e}$.

2.6 *G*-tetrads and *G*-tensors

In the following sections we shall review a number of definitions of Lie derivatives of spinors and related objects, which have appeared in the literature. To this end, we need first to introduce a few preliminary concepts.

Remark 2.6.1 (Terminology). From now on, we shall only consider (classical) G-structures which are "reductive", i.e. where G is a reductive Lie subgroup of $\mathrm{GL}(m, \mathbb{R})$ (*cf.* Definition 2.3.1). On the other hand, note that all the G-structures mentioned in §1.3.1 and encountered so far, i.e. LM, $\mathrm{CSO}(M, g)$ and $\mathrm{SO}(M, g)$, are in fact reductive. Indeed, we know that $\mathrm{SO}(p, q)^e$ is a reductive Lie subgroup of $\mathrm{GL}(M, \mathbb{R})$ (Proposition 2.3.5), and that $\mathrm{GL}(M, \mathbb{R})$ is (trivially) a reductive Lie subgroup of itself. As for $\mathrm{CSO}(p, q)^e$, on recalling that $\mathfrak{cso}(p, q) = \mathfrak{so}(p, q) \oplus \mathbb{R}$ (*cf.* §C.2) and hence noticing that $\mathfrak{gl}(m, \mathbb{R}) = \mathfrak{cso}(p, q) \oplus \mathcal{V}$, where $\mathcal{V}$ is as in Proposition 2.3.5, an argument similar to the one used in proving that proposition shows that $\mathrm{CSO}(p, q)^e$ is indeed a reductive Lie subgroup of $\mathrm{GL}(m, \mathbb{R})$. Also, note that, when considered as a $\mathrm{GL}(m, \mathbb{R})$-structure, LM will be regarded as *gauge*-natural bundle, not just as a (purely) natural one (see, instead, Remark 1.10.11).

Definition 2.6.2. Let $P(M, G)$ be a G-structure on an m-dimensional manifold M and consider the following left action of $W_m^{1,0} G$ on the manifold $\mathrm{GL}(m, \mathbb{R})$

$$\begin{cases} \rho\colon W_m^{1,0} G \times \mathrm{GL}(m, \mathbb{R}) \to \mathrm{GL}(m, \mathbb{R}) \\ \rho\colon ((\alpha^k{}_\ell, a^m{}_n), \beta^i{}_j) \mapsto \beta'^i{}_j := a^i{}_k \beta^k{}_\ell \tilde{\alpha}^\ell{}_j \end{cases} \tag{2.6.1}$$

together with the associated bundle $P_\rho := W^{1,0}P \times_\rho \mathrm{GL}(m, \mathbb{R})$. P_ρ is a fibre bundle associated with $W^{1,0}P$, i.e. a gauge-natural bundle of order $(1, 0)$. A section of P_ρ will be called a ***G-tetrad***[9] (or a ***G-frame***) and usually denoted by θ. Its components will be denoted by $(\theta^a{}_\mu)$ to stress the fact that the second index transforms naturally, whereas the first one, in general, does not. Also, we shall denote by $(e_a{}^\mu)$ the components of the "inverse" of θ, i.e. locally $\theta^a{}_\mu(x)e_a{}^\nu(x) = \delta^\nu{}_\mu$ and $\theta^a{}_\mu(x)e_b{}^\mu(x) = \delta^a{}_b$ for all $x \in M$.

Whenever M admits a spin structure (*cf.* §D.2), the following definition also makes sense.

Definition 2.6.3. Let Λ be the epimorphism which exhibits $\mathrm{Spin}(p, q)^e$ as a two-fold covering of $\mathrm{SO}(p, q)^e$ (*cf.* §D.1) and consider the following left action of $W^{1,0}_m\mathrm{Spin}(p, q)^e$ on $\mathrm{GL}(m, \mathbb{R})$, $p + q = m$ (and m even),

$$\begin{cases} \rho\colon W^{1,0}_m\mathrm{Spin}(p, q)^e \times \mathrm{GL}(m, \mathbb{R}) \to \mathrm{GL}(m, \mathbb{R}) \\ \rho\colon ((\alpha^k{}_\ell, S^m{}_n), \beta^i{}_j) \mapsto \beta'^i{}_j := (\Lambda(S))^i{}_k\beta^k{}_\ell\tilde{\alpha}^\ell{}_j \end{cases}$$

together with the associated bundle $\mathrm{Spin}(M, g)_\rho := W^{1,0}\mathrm{Spin}(M, g)\times_\rho\mathrm{GL}(m, \mathbb{R})$. Clearly, $\mathrm{Spin}(M, g)_\rho$ is a fibre bundle associated with $W^{1,0}\mathrm{Spin}(M, g)$, i.e. a gauge-natural bundle of order $(1, 0)$. A section of $\mathrm{Spin}(M, g)_\rho$ will be called a ***spin-frame-induced tetrad*** or, for short, a ***spin-tetrad***.

In the sequel we shall manly refer to G-tetrads, but it is hereafter understood that, whenever a result holds for an $\mathrm{SO}(p, q)^e$-tetrad (and M admits a spin structure), analogous conclusions can be drawn for a spin-tetrad. Also, one could easily generalize the concept of a G-tetrad to the case of (*general*) *reductive* G-structures, but we shall not need to do so here.

The concept of a G-tetrad also admits the following straightforward generalization.

Definition 2.6.4. Let $P(M, G)$ be a G-structure on M and consider the following left action of the group $W^{1,0}_m G$ on the vector space $T^{r+p}_{s+q}(\mathbb{R}^m)$, $r, p, s, q \in \{0, 1, \ldots, m\}$,

$$\begin{cases} \rho\colon W^{1,0}_m G \times T^{r+p}_{s+q}(\mathbb{R}^m) \to T^{r+p}_{s+q}(\mathbb{R}^m) \\ \rho\colon ((\alpha^g{}_h, a^m{}_n), t^{i_1\ldots i_r k_1\ldots k_p}_{j_1\ldots j_s l_1\ldots l_q}) \mapsto t'^{i_1\ldots i_r k_1\ldots k_p}_{j_1\ldots j_s l_1\ldots l_q} := \\ \quad a^{i_1}{}_{i'_1}\cdots a^{i_r}{}_{i'_r}\alpha^{k_1}{}_{k'_1}\cdots\alpha^{k_p}{}_{k'_p}t^{i'_1\ldots i'_r k'_1\ldots k'_p}_{j'_1\ldots j'_s l'_1\ldots l'_q}(\tilde{a})^{j'_1}{}_{j_1}\cdots(\tilde{a})^{j'_s}{}_{j_s}(\tilde{\alpha})^{l'_1}{}_{l_1}\cdots(\tilde{\alpha})^{l'_q}{}_{l_q}(\det a)^{-v}(\det\alpha)^{-w} \end{cases}$$

together with the associated bundle ${}^{v,w}T^{(r,p)}_{(s,q)}P_\rho := W^{1,0}P \times_\rho T^{r+p}_{s+q}(\mathbb{R}^m)$. ${}^{v,w}T^{(r,p)}_{(s,q)}P_\rho$ is a vector bundle associated with $W^{1,0}P$, i.e. a gauge-natural vector bundle of order $(1, 0)$. A section of ${}^{v,w}T^{(r,p)}_{(s,q)}P_\rho$ will be called a ***G-tensor density field of type*** $\{(r, s), (p, q)\}$ ***and weight*** (v, w) (***on*** P). In particular, a G-tensor density field T of weight $(0, 0)$ will be simply called a ***G-tensor*** (***field***), and its components will be usually denoted by $(T^{a_1\ldots a_r\mu_1\ldots\mu_p}_{b_1\ldots b_s\nu_1\ldots\nu_q})$. Also, a G-tensor field of type $\{(r, s), (0, q)\}$ such that $T^{a_1\ldots a_r}_{b_1\ldots b_s\nu_1\ldots\nu_q} = T^{a_1\ldots a_r}_{b_1\ldots b_s[\nu_1\ldots\nu_q]}$ will be called a ***G-tensor valued*** q***-form***, and a G-tensor of type $\{(1, 0), (0, 0)\}$ will be called a ***G-vector*** (***field***).

[9] The name "tetrad" (in German, "Vierbein") is a slight *abus de langage* here, it not being completely justified but for $m = 4$.

Of course, ${}^{0,0}T^{(1,0)}_{(0,1)}P_\rho \cong P_\rho$ and ${}^{0,w}T^{(0,p)}_{(0,q)}P_\rho \cong {}^{w}T^p_qM$. Thus, a G-tetrad can be regarded as a G-vector valued 1-form, whereas a tensor density field on M of type (p,q) and weight w can be viewed as a G-tensor density field of type $\{(0,0),(p,q)\}$ and weight $(0,w)$. $\mathrm{SO}(1,3)^e$-tensors are particularly important for general relativity, where they are usually known as *Lorentz tensors*.

G-tetrads provide a way to transform G-tensors of type $\{(r,s),(p,q)\}$ to G-tensors of type $\{(r{+}1,s),(p{-}1,q)\}$ and vice versa [and G-tensors of type $\{(r,s),(p,q)\}$ to G-tensors of type $\{(r,s+1),(p,q-1)\}$ and vice versa]. Explicitly,

$$T^{a_1\dots a_r a\dots\mu_p}_{b_1\dots b_s\nu_1\dots\nu_q} := \theta^a{}_{\mu_1}T^{a_1\dots a_r\mu_1\dots\mu_p}_{b_1\dots b_s\nu_1\dots\nu_q}, \quad T'^{\mu\dots a_r\mu_1\dots\mu_p}_{b_1\dots b_s\nu_1\dots\nu_q} := e_{a_1}{}^{\mu}T'^{a_1\dots a_r\mu_1\dots\mu_p}_{b_1\dots b_s\nu_1\dots\nu_q}, \quad \text{etc.} \tag{2.6.2}$$

Remark 2.6.5. Note that, although so far we have adopted either an (abstract) index-free notation or a (concrete) index notation, formulae like the ones above could be interpreted within an "abstract index notation" (*cf.* Penrose & Rindler 1984) since all the objects in question are sections of vector bundles, and hence transform "tensorially". In this framework, Greek and Latin indices would no longer denote holonomic and anholonomic coordinates, respectively, but only natural and gauge (-natural) vector bundle objects. In this context, G-tetrads can be regarded as a sort of "functorial Kronecker δ's".

Now, let $P(M,G)$ be a G-structure, and let '∇' denote the covariant derivative operator associated with the connection on P_ρ induced by a G-connection ω on P and a natural linear connection Γ on LM (*cf.* Examples 1.10.16 and 1.10.17, respectively). Then, we have the following local expression for the covariant derivative of a G-tetrad θ:

$$\nabla_\mu\theta^a{}_\nu = \partial_\mu\theta^a{}_\nu + \omega^a{}_{b\mu}\theta^b{}_\nu - \Gamma^\rho{}_{\nu\mu}\theta^a{}_\rho.$$

In the sequel, we shall always require that θ satisfy the *compatibility condition* (*with* ∇), i.e.

$$\nabla\theta = 0, \tag{2.6.3}$$

which locally implies

$$\omega^a{}_{b\mu} = \theta^a{}_\nu\partial_\mu e_b{}^\nu + \theta^a{}_\rho\Gamma^\rho{}_{\nu\mu}e_b{}^\nu, \tag{2.6.4}$$

formally identical with (1.5.15), but here the $\omega^a{}_{b\mu}$'s are *not* the $\Gamma^\rho{}_{\nu\mu}$'s in anholonomic coordinates, but the components of a G-connection ω on P, *a priori* unrelated to the linear connection Γ on LM.

In the case $P = \mathrm{SO}(M,g)$, we have defined a metric tensor g on M, and let $(g_{\mu\nu})$ be its components in some natural chart. Then, we can define a (non-degenerate symmetric) $\mathrm{SO}(p,q)^e$-tensor $\hat g$ of type $\{(0,2),(0,0)\}$ by

$$g_{ab} = e_a{}^\mu e_b{}^\nu g_{\mu\nu}, \tag{2.6.5}$$

g_{ab} denoting the components of $\hat g$ is some suitable basis [equivalently, one could interpret (2.6.5) abstractly]. Now we know that $\hat g$ must transform as an $\mathrm{SO}(p,q)^e$-tensor of type $\{(0,2),(0,0)\}$: hence,

$$g_{ab} \mapsto g'_{ab} = L^c{}_aL^d{}_bg_{cd} \tag{2.6.6}$$

for some $L \in \mathrm{SO}(p,q)^e$. But then we can always choose coordinates such that $g_{ab} = \eta_{ab}$, where η is the standard Minkowski metric defined in §C.1: in the sequel, we shall always assume that this choice has been made, i.e.

$$\eta_{ab} = e_a{}^\mu e_b{}^\nu g_{\mu\nu}, \tag{2.6.7}$$

the inverse transformation being of course

$$g_{\mu\nu} = \theta^a{}_\mu \theta^b{}_\nu \eta_{ab}. \tag{2.6.8}$$

In Chapter 4 we shall adopt a different view, where the metric g is regarded as being determined *a posteriori* via (2.6.8) by a given (*free*) spin-tetrad, the latter being in turn determined by Einstein's field equations.

Remark 2.6.6. Of course, given a metric tensor g on M, we can always construct a (non-degenerate symmetric) G-tensor $\hat{g}$ via (2.6.5) even for a generic G-structure $P(M, G)$, the inverse transformation being

$$g_{\mu\nu} = \theta^a{}_\mu \theta^b{}_\nu g_{ab}, \tag{2.6.9}$$

but now, since (2.6.6) will hold for some $L \in G$ with $G \neq \mathrm{SO}(p, q)^e$ (in general), then we cannot make the predefined choice $g_{ab} = \eta_{ab}$.

Remark 2.6.7. From now on, G-tensor Greek [Latin] indices are assumed to be lowered by $g_{\mu\nu}$ [g_{ab}] and raised by $g^{\nu\rho}$ [g^{bc}], $g^{\nu\rho}$ [g^{bc}] denoting the inverse of $g_{\mu\nu}$ [g_{ab}], and we shall use the same kernel letter as in (2.6.2). Then, one readily verifies that

$$\theta_{a\mu} = e_{a\mu} \qquad \text{and, equivalently,} \qquad e^{b\mu} = \theta^{b\mu}. \tag{2.6.10}$$

Note that, in the case of an $\mathrm{SO}(p, q)^e$-structure, compatibility condition (2.6.3) implies metricity condition (1.5.19) [*cf.* (2.6.8)], whereas for a generic G-structure this is not necessarily true since, in general, $\nabla\hat{g} \neq 0$ [*cf.* (2.6.9)]. Throughout this thesis, though, we shall always assume metricity, i.e.

$$\nabla g = 0, \tag{2.6.11}$$

whence also $\nabla\hat{g} = 0$. The fact that the compatibility condition for an $\mathrm{SO}(p, q)^e$-tetrad is equivalent to the metricity of Γ is consistent with the standard result saying that a metric linear connection can always be reduced to a principal connection on $\mathrm{SO}(M, g)$ [*cf.* (2.6.4)].

Of course, we can easily compute the gauge-natural Lie derivative of a G-tensor with respect to a G-invariant vector field Ξ on P projecting on a vector field ξ on M [*cf.* (1.3.10)]. E.g., for a G-vector field η, it locally reads [*cf.* (2.1.10)]

$$\pounds_\Xi \eta^a = \xi^\mu \partial_\mu \eta^a - \Xi^a{}_b \eta^b \tag{2.6.12}$$

and, for a G-tetrad θ,

$$\pounds_\Xi \theta^a{}_\mu = \xi^\nu \partial_\nu \theta^a{}_\mu - \Xi^a{}_b \theta^b{}_\mu + \partial_\mu \xi^\nu \theta^a{}_\nu. \tag{2.6.13}$$

In the sequel, it will be often convenient to express such Lie derivatives in a "covariantized form" in order to exploit conditions like (2.6.3) or (2.6.11). To do so, recall first the local expression (1.5.4) of the vertical part of a G-invariant vector field on a principal bundle P.

Specialized to the case of a G-structure, this reads

$$\check{\Xi} = (\Xi^a{}_b + \omega^a{}_{b\mu}\xi^\mu)\rho_a{}^b. \tag{2.6.14}$$

On making use of this expression it is easy to see that, e.g., (2.6.12) and (2.6.13) can be recast into the form

$$\pounds_\Xi \eta^a = \xi^\mu \nabla_\mu \eta^a - \check{\Xi}^a{}_b \eta^b \tag{2.6.12$'$}$$

and

$$\pounds_\Xi \theta^a{}_\mu = \tilde{\nabla}_\mu \xi^\nu \theta^a{}_\nu - \check{\Xi}^a{}_b \theta^b{}_\mu, \tag{2.6.13$'$}$$

respectively, where in the latter we also took (2.6.3) into account. In this form, these expressions lend themselves to a reinterpretation in terms of abstract index notation, since the vertical part of a vector field transforms tensorially [*cf.* (1.3.13)], and G-tensors are sections of (gauge-natural) vector bundles (*cf.* Remark 2.6.5 above).

Now we would like to compute the G-reductive Lie derivative of a G-tensor: we can do this directly for G-tensors of type $\{(r,s),(0,0)\}$, for which it coincides with the gauge-natural one (*cf.* Remark 2.5.15). In order to compute the G-reductive Lie derivative of a G-tensor of arbitrary type, we first need to regard $W^{1,0}P$ as $W^{0,0}(W^{1,0}P)$, i.e. to think of LM as a generic gauge-natural bundle rather than a purely natural one (*cf.* Proposition 2.5.17 and Example 2.5.19). Then, as we do not want to consider a generic $W^{1,0}_m G$-invariant vector field on $W^{1,0}P$, but one constructed out of a $\mathrm{GL}(m,\mathbb{R})$-invariant vector field Ξ on LM projecting on a vector field ξ on M, we shall define

$$\pounds^G_\Xi T := \pounds^{W^{1,0}_m G}_{\Xi_\mathrm{K}\oplus_\xi \Xi_\mathrm{K}} T,$$

where $\Xi_\mathrm{K} \oplus_\xi \Xi_\mathrm{K}$ is the $W^{1,0}_m G$-invariant vector field on $W^{1,0}P$ constructed out of the Kosmann vector field Ξ_K on P associated with Ξ. The definition makes sense because Ξ_K is a G-invariant vector field on P, but is $\mathrm{GL}(m,\mathbb{R})$-invariant when regarded as a vector field on the corresponding subset of LM (*cf.* Remark 2.4.8). If

$$\Xi_\mathrm{K} = \xi^\mu \partial_\mu + (\Xi_\mathrm{K})^a{}_b \rho_a{}^b$$

is the local expression of Ξ_K on P, the local expression of $\Xi_\mathrm{K} \oplus_\xi \Xi_\mathrm{K}$ is given by

$$\Xi_\mathrm{K} \oplus_\xi \Xi_\mathrm{K} = \xi^\mu \partial_\mu + (\Xi_\mathrm{K})^\mu{}_\nu \rho_\mu{}^\nu + (\Xi_\mathrm{K})^a{}_b \rho_a{}^b,$$

where $(\rho_\mu{}^\nu)$ $[(\rho_a{}^b)]$ is a local basis of $\mathrm{GL}(m,\mathbb{R})$-invariant [$G$-invariant] vector fields on LM [P]. Of course, the $(\Xi_\mathrm{K})^\mu{}_\nu$'s are related to the $(\Xi_\mathrm{K})^a{}_b$'s via (1.3.14) and (1.3.14$'$), i.e.

$$(\Xi_\mathrm{K})^a{}_b = e_b{}^\nu \xi^\mu \partial_\mu \theta^a{}_\nu + \theta^a{}_\mu (\Xi_\mathrm{K})^\mu{}_\nu e_b{}^\nu \tag{2.6.15}$$

and

$$(\Xi_\mathrm{K})^\mu{}_\nu = \theta^a{}_\nu \xi^\mu \partial_\mu e_a{}^\nu + e_a{}^\mu (\Xi_\mathrm{K})^a{}_b \theta^b{}_\nu, \tag{2.6.15$'$}$$

but here the $\theta^a{}_\mu$'s [$e_a{}^\mu$'s] must be regarded as the components of [the inverse of] a G-tetrad, transforming a G-invariant vector field on P into a $\mathrm{GL}(m,\mathbb{R})$-invariant vector field on LM. So, we see, for instance, that the local expression of the $\mathrm{GL}(m,\mathbb{R})$-reductive Lie derivative of a vector field η on M with respect to a $\mathrm{GL}(m,\mathbb{R})$-invariant vector field

$\Xi = \Xi_{\rm K}$ on LM is

$$\pounds_{\Xi}^{\mathrm{GL}(m,\mathbb{R})} \eta^\mu = \xi^\nu \partial_\nu \eta^\mu - \Xi^\mu{}_\nu \eta^\nu,$$

which coincides, of course, with (the local expression of) its gauge-natural Lie derivative $\pounds_\Xi \eta$, *provided* we consider η as a *gauge*-natural, not simply natural, object (*cf.* Remark 2.1.5). We already had an example of this behaviour in Example 2.5.19: natural object are transformed in a *gauge*-natural way rather than in the usual natural way. This is also why we questioned the applicability of reductive Lie derivatives to concrete physical situations in which the given physical entity happens to be represented by a natural object. Also, it should not go unnoticed that the G-reductive Lie derivative of a G-tetrad is identically zero. Indeed,

$$\pounds_{\Xi}^{G} \theta^a{}_\mu = \xi^\nu \partial_\mu \theta^a{}_\nu - (\Xi_{\rm K})^a{}_b \theta^b{}_\mu + (\Xi_{\rm K})^\nu{}_\mu \theta^a{}_\nu = 0$$

by (2.6.15).

We conclude by giving a coordinate (or abstract index) proof of Corollary 2.5.18. Let then $\check{\Xi}_{\rm K}$ be the vertical part of the Kosmann vector field associated with a $\mathrm{GL}(m,\mathbb{R})$-invariant vector field Ξ on LM with respect to some metric linear connection, e.g. the Levi-Civita connection associated with g. In covariantized form and using (2.6.11), (1.3.13) and (2.6.10)(2), we obtain

$$\pounds_{\Xi}^{\mathrm{SO}(p,q)^e} g_{\mu\nu} = 2(\check{\Xi}_{\rm K})_{(\mu\nu)} = 2\theta^a{}_\mu (\check{\Xi}_{\rm K})_{(ab)} \theta^b{}_\nu,$$

but $(\check{\Xi}_{\rm K})_{ab} = (\check{\Xi}_{\rm K})_{[ab]}$ because $\Xi_{\rm K}$ is $\mathrm{SO}(p,q)^e$-invariant, whence the result follows. □

2.6.1 Holonomic gauge

We have just seen that the *G-reductive* Lie derivative of a G-tetrad is identically zero. Consider now the *gauge-natural* Lie derivative of a $\mathrm{GL}(m,\mathbb{R})$-tetrad with respect to the natural lift $L\xi$ onto LM of a vector field ξ on M. On using (2.4.6), we readily get

$$\pounds_{L\xi} \theta = 0.$$

This is obvious because, in this particular case, the $\mathrm{GL}(m,\mathbb{R})$-invariant vector field on LM with respect to which the Lie derivative is taken is required to coincide with the natural lift of ξ. Accordingly,

$$\begin{aligned} \pounds_{L\xi} T^{a\ldots\mu\ldots}_{b\ldots\nu\ldots} &\equiv \pounds_{L\xi} (\theta^a{}_\rho \cdots e_b{}^\sigma \cdots T^{\rho\ldots\mu\ldots}_{\sigma\ldots\nu\ldots}) \\ &\equiv \theta^a{}_\rho \cdots e_b{}^\sigma \cdots \pounds_{L\xi} T^{\rho\ldots\mu\ldots}_{\sigma\ldots\nu\ldots} \\ &\equiv \theta^a{}_\rho \cdots e_b{}^\sigma \cdots \pounds_{\xi} T^{\rho\ldots\mu\ldots}_{\sigma\ldots\nu\ldots} \end{aligned} \tag{2.6.16}$$

for any $\mathrm{GL}(m,\mathbb{R})$-tensor T with components $(T^{\mu\ldots a\ldots}_{\nu\ldots b\ldots})$. Identity (2.6.16) is a sort of "naturality condition", since applying $\pounds_{L\xi}$ to a section of $T^{(r,p)}_{(s,q)} LM_\rho$ turns out to be the same as applying the ordinary (i.e. natural) Lie derivative to the corresponding section of $T^{(0,r+p)}_{(0,s+q)} LM_\rho \cong T^{r+p}_{s+q} M$ and then transvecting the result with as many $\mathrm{GL}(m,\mathbb{R})$-tetrads as necessary to map it back to $T^{(r,p)}_{(s,q)}LM_\rho$. Vice versa, note that, if we require

$$\pounds_\Xi \theta = 0 \tag{2.6.17}$$

for a $\mathrm{GL}(m,\mathbb{R})$-tetrad θ and a *generic* $\mathrm{GL}(m,\mathbb{R})$-invariant vector field Ξ on LM projecting on a vector field ξ on M, on using (2.6.3) and recalling (2.4.6), we find precisely

$$\check{\Xi}^a{}_b = \tilde{\nabla}_b \xi^a \equiv (\check{L\xi})^a{}_b \quad \text{i.e.} \quad \Xi = L\xi, \tag{2.6.18}$$

where the use of the transposed linear connection indicates that, for the moment, we are allowing for non-symmetric (metric) connections.

Therefore, we can ask ourselves what happens if we require (2.6.17) to hold for any G-tetrad and any G-invariant vector field Ξ on a generic G-structure P. In line with (2.6.16), we shall call condition (2.6.17) the ***holonomic gauge*** or, for reasons which will become apparent in a moment, the ***G-Killing equation***[10].

Now, on using (2.6.13′) and recalling (2.4.7), it is almost immediate to realize that, if condition (2.6.17) is imposed, then

$$\check{\Xi}_{ab} = -\tilde{\nabla}_{[a}\xi_{b]} \equiv (\check{L\xi})_{[ab]} \equiv (\check{\xi}_{\mathrm{K}})_{ab} \quad \text{i.e.} \quad \Xi = \xi_{\mathrm{K}}.$$

But now, unlike in the G-reductive case, $\pounds_{\xi_{\mathrm{K}}}$ must reduce to $\pounds_\xi$ on natural objects. Therefore, from (2.6.17) and (2.6.8) we automatically get

$$\pounds_\xi g_{\mu\nu} \equiv \pounds_{\xi_{\mathrm{K}}} g_{\mu\nu} \equiv 0, \tag{2.6.19}$$

i.e. *imposing the holonomic gauge on an* $\mathrm{SO}(p,q)^e$*-tetrad amounts to setting* $\Xi = \xi_{\mathrm{K}}$ and *requiring* ξ *to be* Killing (*cf.* §B.1). Similarly, in the case of a $\mathrm{CSO}(p,q)^e$-structure, we find

$$\check{\Xi}_{ab} = -\tilde{\nabla}_{[a}\xi_{b]} + \frac{1}{m}\tilde{\nabla}_c\xi^c g_{ab} \equiv (\check{L\xi})_{[ab]} + \frac{1}{m}(\check{L\xi})^c{}_c g_{ab} \equiv (\check{\xi}_{\mathrm{P}})_{ab} \quad \text{i.e.} \quad \Xi = \xi_{\mathrm{P}},$$

where ξ_{P} is the generalized Kosmann lift of ξ onto $\mathrm{CSO}(M,g)$, called, for reasons that will become apparent in the sequel, the *Penrose lift* (*of* ξ). Also, from (2.6.17) and (2.6.9) and using the fact that $\Xi = \xi_{\mathrm{P}}$ [or simply recalling that Ξ is $\mathrm{CSO}(p,q)^e$-invariant][11],

$$\pounds_\xi g_{\mu\nu} \equiv \pounds_{\xi_{\mathrm{P}}} g_{\mu\nu} \equiv \frac{2}{m}\tilde{\nabla}_\rho \xi^\rho g_{\mu\nu},$$

where, as before, '$\tilde{\nabla}$' denotes the covariant derivative associated with the transpose of a (not necessarily symmetric) metric linear connection. Thus, *imposing the holonomic gauge on a* $\mathrm{CSO}(p,q)^e$*-tetrad amounts to setting* $\Xi = \xi_{\mathrm{P}}$ and *requiring* ξ *to be* conformal Killing (*cf.*, again, §B.1).

[10] This name was suggested by Marco Godina.

[11] In the same way we could have proven (2.6.19), i.e. *without* assuming (2.6.8).

2.6.2 Lie derivative of a G-connection

Although a G-connection on a principal bundle $P(M,G)$ is *not* a G-tensor even when P is a G-structure since it is section of a gauge-natural *affine* bundle (*cf.* Example 1.10.16), we shall give here the local expression of its Lie derivative for future reference. We shall start with the case of a generic principal bundle, and then specialize to the case of G-structures.

So, let Ξ be a G-invariant vector field on P locally reading

$$\Xi(x,a) = \xi^\mu(x)\partial_\mu + \Xi^{\mathcal{A}}(x)\rho_{\mathcal{A}}(a)$$

for all $\psi_\alpha^{-1}(x,a) \in P$, projecting on a vector field ξ on M, locally reading $\xi(x) = \xi^\mu(x)\partial_\mu$ for all $x \in M$. Then, if $P_\ell := W^{1,1}P \times_\ell \mathcal{A}$ denotes the bundle of G-connections on P (*cf.* Example 1.10.16), the induced vector field $\Xi_\ell \in \mathfrak{X}(P_\ell)$ will locally read

$$\Xi_\ell = \xi^\mu\partial_\mu + \Xi^{\mathcal{A}}{}_\mu \frac{\partial}{\partial w^{\mathcal{A}}{}_\mu},$$

where

$$\Xi^{\mathcal{A}}{}_\mu = -(\partial_\mu \xi^\nu w^{\mathcal{A}}{}_\nu - (\mathrm{ad}_{\Xi_e})^{\mathcal{A}}{}_{\mathcal{B}} w^{\mathcal{B}}{}_\nu + \partial_\mu \Xi^{\mathcal{A}}), \tag{2.6.20}$$

$(w^{\mathcal{A}}{}_\mu)$ are local fibre coordinates on P_ℓ and $\Xi_e(x) \equiv \Xi^{\mathcal{A}}(x)\varepsilon_{\mathcal{A}}$, $(\varepsilon_{\mathcal{A}})$ being a basis of $\mathfrak{g}$. Indeed, we have

$$\begin{aligned} \xi^\lambda(x) &= \tfrac{\partial}{\partial t}\varphi_t^\lambda(x)\Big|_{t=0}, \\ \Xi^{\mathcal{A}}(x) &= \tfrac{\partial}{\partial t} f_t^{\mathcal{A}}(x)\Big|_{t=0}, \\ \Xi^{\mathcal{A}}{}_\mu(x) &= \tfrac{\partial}{\partial t}\Big((\Phi_t)_\lambda\Big)^{\mathcal{A}}{}_\mu(x)\Big|_{t=0}, \end{aligned}$$

$\{\Phi_t\}$ denoting the flow of Ξ (*cf.* §1.3). Hence, setting $\alpha^\nu{}_\mu(x) = \partial\varphi_t^\nu(x)/\partial x^\mu$, $a(x) = \Phi_t(x)$ and $\omega'^{\mathcal{A}}{}_\mu(x) := w'^{\mathcal{A}}{}_\mu \circ \omega(x) = \Big((\Phi_t)_\lambda\Big)^{\mathcal{A}}{}_\mu(x)$ for any section ω of P_ℓ, we can rewrite (1.10.7) as

$$\Big((\Phi_t)_\lambda\Big)^{\mathcal{A}}{}_\mu(x) = \Big((\mathrm{Ad}_{\Phi_t(x)})^{\mathcal{A}}{}_{\mathcal{B}}\omega^{\mathcal{B}}{}_\nu(x) - (\partial_\nu\Phi_t \circ \Phi_{-t}(x))^{\mathcal{A}}\Big)\frac{\partial\varphi^\nu_{-t}(x)}{\partial x^\mu}.$$

Differentiating this equation with respect to t at $t=0$ then gives

$$\Xi^{\mathcal{A}}{}_\mu(x) = (\mathrm{ad}_{\Xi_e(x)})^{\mathcal{A}}{}_{\mathcal{B}}\omega^{\mathcal{B}}{}_\mu(x) - \partial_\mu\Xi^{\mathcal{A}}(x) - \omega^{\mathcal{A}}{}_\nu(x)\partial_\mu\xi^\nu(x),$$

i.e. precisely (2.6.20). □

Therefore, by virtue of (2.1.8) and (2.6.20) the Lie derivative of a G-connection ω is

$$\pounds_\Xi \omega^{\mathcal{A}}{}_\mu = \xi^\nu\partial_\nu\omega^{\mathcal{A}}{}_\mu + \partial_\mu\xi^\nu\omega^{\mathcal{A}}{}_\nu - (\mathrm{ad}_{\Xi_e})^{\mathcal{A}}{}_{\mathcal{B}}\omega^{\mathcal{B}}{}_\mu + \partial_\mu\Xi^{\mathcal{A}}$$

or

$$\pounds_\Xi \omega^{\mathcal{A}}{}_\mu = \xi^\nu\partial_\nu\omega^{\mathcal{A}}{}_\mu + \partial_\mu\xi^\nu\omega^{\mathcal{A}}{}_\nu - c^{\mathcal{A}}{}_{\mathcal{BC}}\,\Xi^{\mathcal{B}}\omega^{\mathcal{C}}{}_\mu + \partial_\mu\Xi^{\mathcal{A}}, \tag{2.6.22}$$

where we introduced the *structure constants* ($c^{\mathcal{A}}{}_{\mathcal{BC}} \equiv c^{\mathcal{A}}{}_{[\mathcal{BC}]}$) of the Lie algebra $\mathfrak{g}$ [with respect to the basis $(\varepsilon_{\mathcal{A}})$], intrinsically defined by $\mathrm{ad}_{\varepsilon_{\mathcal{B}}}\varepsilon_{\mathcal{C}} \equiv [\varepsilon_{\mathcal{B}}, \varepsilon_{\mathcal{A}}] = c^{\mathcal{A}}{}_{\mathcal{BC}}\,\varepsilon_{\mathcal{A}}$ (*cf.* §C.2).

On a G-structure, formula (2.6.22) becomes

$$\pounds_{\Xi}\omega^a{}_{b\mu} = \xi^\nu \partial_\nu \omega^a{}_{b\mu} + \partial_\mu \xi^\nu \omega^a{}_{b\nu} + \omega^a{}_{c\mu}\Xi^c{}_b - \omega^c{}_{b\mu}\Xi^a{}_c + \partial_\mu \Xi^a{}_b \tag{2.6.23}$$

in the fundamental representation of the Lie algebra $\mathfrak{g}$ of G on $\mathfrak{gl}(m,\mathbb{R})$.

Now we know that, P_ℓ being an affine bundle, $\pounds_\Xi \omega$ must be a section of the associated vector bundle $\vec{P}_\ell$ (*cf.* Remark 2.1.3). Therefore, we would like to recast (2.6.22) and (2.6.23) into a form which made their tensorial character explicit.

To this end, first note that the bundle of vertical G-invariant vector fields is isomorphic to the vector bundle $P \times_{\mathrm{Ad}} \mathfrak{g}$ (*cf.* Example 1.10.19). Define then the $(P \times_{\mathrm{Ad}} \mathfrak{g})$-valued ***curvature* 2*-form*** Ω associated with ω by

$$\Omega(\xi,\eta) = \widehat{[\xi,\eta]} - [\hat{\xi},\hat{\eta}] \tag{2.6.24}$$

for all $\xi,\eta \in \mathfrak{X}(M)$, $\hat{\xi}$ and $\hat{\eta}$ being the horizontal lifts of ξ and η with respect to ω, respectively. Locally,

$$\Omega^{\mathcal{A}}{}_{\mu\nu} = \partial_\mu \omega^{\mathcal{A}}{}_\nu - \partial_\nu \omega^{\mathcal{A}}{}_\mu + c^{\mathcal{A}}{}_{\mathcal{BC}}\, \omega^{\mathcal{B}}{}_\mu \omega^{\mathcal{C}}{}_\nu. \tag{2.6.25}$$

Now, since $(P \times_{\mathrm{Ad}} \mathfrak{g})$ is a vector bundle associated with P, it makes sense to consider covariant exterior derivatives of $(P \times_{\mathrm{Ad}} \mathfrak{g})$-valued p-forms (*cf.* §1.5). In particular,

$$\mathcal{D}_\mu \check{\Xi}^{\mathcal{A}} = \partial_\mu \check{\Xi}^{\mathcal{A}} + c^{\mathcal{A}}{}_{\mathcal{BC}}\, \omega^{\mathcal{B}}{}_\mu \check{\Xi}^{\mathcal{C}} = \nabla_\mu \check{\Xi}^{\mathcal{A}}. \tag{2.6.26}$$

Hence, on using (2.6.23) and (1.5.4), we easily find the desired expression

$$\pounds_\Xi \omega^{\mathcal{A}}{}_\mu = \xi^\nu \Omega^{\mathcal{A}}{}_{\nu\mu} + \nabla_\mu \check{\Xi}^{\mathcal{A}} \tag{2.6.27}$$

for the Lie derivative of a G-connection on a principal bundle $P(M,G)$. Analogously, on using (2.6.23) and (2.6.14), it is easy to verify that

$$\pounds_\Xi \omega^a{}_b = \xi \,\lrcorner\, \Omega^a{}_b + \mathcal{D}\check{\Xi}^a{}_b, \tag{2.6.28}$$

$\pounds_\Xi \omega^a{}_b := \pounds_\Xi \omega^a{}_{b\mu}\, \mathrm{d}x^\mu$, $\Omega^a{}_b := 1/2\, \Omega^a{}_{b\sigma\mu}\, \mathrm{d}x^\sigma \wedge \mathrm{d}x^\mu$, or equivalently

$$\pounds_\Xi \omega^a{}_{b\mu} = \theta^a{}_\rho e_b{}^\nu R^\rho{}_{\nu\sigma\mu} \xi^\sigma + \nabla_\mu \check{\Xi}^a{}_b, \tag{2.6.28$'$}$$

where we used the fact that $\Omega^a{}_{b\sigma\mu} \equiv \theta^a{}_\rho e_b{}^\nu R^\rho{}_{\nu\sigma\mu}$, $(\Omega^a{}_{b\sigma\mu})$ being the components of Ω and $(R^\rho{}_{\nu\sigma\mu})$ the components of the curvature 2-form[12] associated with the (compatible) natural linear connection Γ: the two sets are related because of (2.6.4). On starting from (2.6.28$'$), a rather lengthy but straightforward calculation, which makes use of (2.6.4), (2.1.7), (2.6.3) and (2.6.14), gives

$$\pounds_\xi \Gamma^\rho{}_{\nu\mu} \equiv \pounds_\Xi \Gamma^\rho{}_{\nu\mu} = R^\rho{}_{\nu\sigma\mu}\xi^\sigma + \nabla_\mu \tilde{\nabla}_\nu \xi^\rho, \tag{2.6.29}$$

a formula that has been known for a long time (*cf.*, e.g., Schouten 1954; Yano 1957), and

[12]The corresponding tensor of type $(1,3)$ is, of course, the well-known *Riemann tensor* (of M: *cf.*, e.g., Wald 1984).

is usually obtained directly from (1.5.16) or (1.10.8) on applying the standard definition of a Lie derivative of *natural* objects.

Now, recall that the conventional notations $\pounds_{\Xi}\omega^a{}_{b\mu}$ and $\pounds_{\xi}\Gamma^\rho{}_{\nu\mu}$ actually stand for $(\pounds_{\Xi}\omega)^a{}_{b\mu}$ and $(\pounds_{\xi}\Gamma)^\rho{}_{\nu\mu}$, respectively, where Γ is a *natural* linear connection (*cf.* Example 1.10.17). Of course, $\omega^\rho{}_{\nu\mu} \equiv \Gamma^\rho{}_{\nu\mu}$, i.e. the components of ω with respect to a holonomic basis coincide with the components of Γ with respect to the same basis [*cf.* (2.6.4)], but in general $\pounds_{\Xi}\omega^\rho{}_{\nu\mu} \neq \pounds_{\xi}\Gamma^\rho{}_{\nu\mu}$, i.e. the components of $\pounds_{\Xi}\omega$ with respect to a holonomic basis do not coincide, in general, with the components of $\pounds_{\xi}\Gamma$ with respect to the same basis. Until the general theory of Lie derivatives was developed, there lacked the mathematical framework for evaluating the Lie derivative of a (general) G-connection, and it was only possible to evaluate $\pounds_{\xi}\Gamma$. Classical textbooks do sometimes give the expression $\pounds_{\xi}\Gamma^a{}_{b\mu}$ for the Lie derivative of a natural linear connection in anholonomic coordinates, but this is of course different from $\pounds_{\Xi}\omega^a{}_{b\mu}$ *even* when ω is a $\mathrm{GL}(m, \mathbb{R})$-connection on LM since the morphisms are different (*cf.* Example 1.10.17).

Formula (2.6.28) will play a key role in the derivation of the gravitational superpotential in Chapter 4 in exactly the same way as formula (2.6.29) plays a key role in the previous purely natural formulations of gravity (*cf.* §3.3.5). As the primary object of our gauge-natural gravitation theory will be a spin-tetrad, we will be forced to take (2.6.28) instead of (2.6.29) as the appropriate formula for the Lie derivative of a connection, whether this is regarded as a variable itself (as in a "metric-affine" formulation) or a combination of spin-tetrads (as in a purely "metric" one).

2.7 Critical review of some Lie derivatives

As anticipated in §2.6, we shall now review some definitions of Lie derivatives of spinor fields and G-tensors which have appeared in the literature in recent years, one of the most noteworthy being of course Bourguignon & Gauduchon's (1992), which has been already analysed in §2.5. All Lie derivatives reviewed in this section are of the gauge-natural, not the G-reductive type, since all the authors under consideration aim at defining operators which suitably reduce to the standard one on natural objects.

2.7.1 Penrose's Lie derivative of "spinor fields"

We shall now give a reinterpretation of Penrose & Rindler's (1986) definition of a Lie derivative of spinor fields in the light of the general theory of Lie derivatives. This definition has become quite popular among the physics community despite its being restricted to infinitesimal conformal isometries, and was already thoroughly analysed by Delaney (1993). Although his analysis is correct for all practical purposes, Delaney fails to understand the true reasons of the aforementioned restriction because these are, crucially, of a functorial nature, and only a functorial analysis can unveil them.

Throughout this section we shall assume $m \equiv \dim M = 4$, and '∇' will denote the covariant derivative operator corresponding to the Levi-Civita connection associated with the given metric g. Also, we shall use a covariantized form for all our Lie derivatives so that our formulae can be reinterpreted in an abstract index fashion, if so desired. We refer the reader to §D.3 for the basics of the 2-spinor formalism.

Penrose & Rindler's Lie derivative $\mathcal{L}^{\mathrm{P}}_{\xi}\phi$ of a 2-spinor field ϕ (locally) reads

$$\mathcal{L}^{\mathrm{P}}_{\xi}\phi^A = \xi^a\nabla_a\phi^A - \left(\frac{1}{2}\nabla_{BA'}\xi^{AA'} + \frac{1}{4}\nabla_c\xi^c\delta^A{}_B\right)\phi^B, \tag{2.7.1}$$

(*cf.* Penrose & Rindler 1986, §6.6) or, in 4-spinor formalism,

$$\mathcal{L}^{\mathrm{P}}_{\xi}\psi = \xi^a\nabla_a\psi - \left(\frac{1}{4}\nabla_{[a}\xi_{b]}\gamma^a\gamma^b + \frac{1}{4}\nabla_c\xi^c\right)\psi \tag{2.7.2}$$

for some suitable 4-spinor ψ, and, in the authors' formulation, only holds if ξ is conformal Killing. From (2.7.2) we immediately note that the (particular) lift of ξ with respect to which the Lie derivative is taken is *not* $\mathrm{SO}(p,q)^e$-invariant, but rather $\mathrm{CSO}(p,q)^e$-invariant, i.e., strictly speaking $\mathcal{L}^{\mathrm{P}}_{\xi}\psi$ is not a Lie derivative of a spinor field, but of a *conformal* spinor field. Furthermore, it is easy to realize that the given lift is actually the Penrose lift defined in §2.6.1.

Now, the Penrose lift ξ_{P} of a vector field ξ is just a particular instance of a generalized Kosmann lift, and one should be able to take the Lie derivative of a conformal spinor field with respect to ξ_{P} no matter what ξ is (*cf.* Definition 2.5.2 and Example 2.5.3). The fact that formula (2.7.1) above is stated to hold only for conformal Killing vector fields then suggests that an additional condition has been (tacitly) imposed. From the discussion in §2.6.1 we deduce that this condition must be the holonomic gauge.

This is actually the impression one gets from Huggett & Tod (1994), who from the 2-dimensionality of $S(M,g)$ deduce that, for *all* Lie derivatives $\mathcal{L}_{\Xi}$ of spinor fields, the following should hold

$$\mathcal{L}_{\Xi}g_{ab} \equiv \mathcal{L}_{\Xi}(\varepsilon_{AB}\varepsilon_{A'B'}) = (\lambda + \bar{\lambda})g_{ab}. \tag{2.7.3}$$

This is true: actually, in the strictly spinorial case $\lambda = 0$ since $g_{ab} = \eta_{ab}$, but what is not necessarily true is that

$$\mathcal{L}_{\Xi}g_{\mu\nu} \equiv \mathcal{L}_{\xi}(\theta^a{}_\mu\theta^b{}_\nu g_{ab}) = (\lambda + \bar{\lambda})g_{\mu\nu},$$

unless we impose precisely the holonomic gauge[13]: this implies nice property (2.6.16), but has the strong drawback of restricting ourselves to infinitesimal conformal isometries. This is probably the simplest way to interpret Penrose's Lie derivative in terms of the general theory of Lie derivatives.

This is not, though, the way Penrose & Rindler (1986) arrive at formula (2.7.1). Using the fact that a (complex) bivector field K, i.e. a section of $\bigwedge^2 TM^{\mathbb{C}}$, can be represented spinorially as $\kappa\otimes\kappa\otimes\bar{\varepsilon}$, κ being a section of $S(M,g)$, they write

$$\mathcal{L}^{\mathrm{P}}_{\xi}(\kappa^A\kappa^B\varepsilon^{A'B'}) = \xi^c\nabla_c(\kappa^A\kappa^B\varepsilon^{A'B'}) - \kappa^D\kappa^B\varepsilon^{D'B'}\nabla_d\xi^a - \kappa^A\kappa^D\varepsilon^{A'D'}\nabla_d\xi^b. \tag{2.7.4}$$

assuming that $\kappa\otimes\kappa\otimes\bar{\varepsilon}$ transforms with the usual natural lift. In §D.3, though, we mentioned that, in this kind of vector bundle isomorphisms, $TM^{\mathbb{C}}$ (or TM) is assumed

[13]Note that, even in an abstract index type notation, we *cannot* suppress the $\mathrm{CSO}(p,q)^e$-tetrads since the Lie derivative is a category-dependent operator, and the $\mathrm{CSO}(p,q)^e$-tetrads here serve the purpose of reminding us that, ultimately, we still want g to be a natural object as far as Lie differentiation is concerned.

to be a *gauge*-natural vector bundle associated with $\mathrm{SO}(M, g)$, *not* a natural bundle (associated with LM). Hence, if we want (2.7.4) to make any sense at all from the point of view of the general theory of Lie derivatives, we must interpret $\pounds_\xi^{\mathrm{P}} K$ as a $\pounds_{L\xi} K$ on a vector bundle associated with a $\mathrm{GL}(4, \mathbb{R})$-structure, not as $\pounds_\xi K$: in any case, κ will not transform with $\mathrm{SL}(2, \mathbb{C})$, as a standard spinor.

Now, a little algebra (Penrose & Rindler 1986, p. 102) shows that (2.7.4) implies

$$\nabla_{(A}^{(A'} \xi_{B)}^{B')} = 0, \tag{2.7.5}$$

which is readily seen to be equivalent to the conformal Killing equation (*cf.* §B.1). But now recall from §2.6.1 that $\pounds_{L\xi} K$ identically satisfies the holonomic gauge, and, unlike in the $\mathrm{SO}(p, q)^e$- or the $\mathrm{CSO}(p, q)^e$-case, this did not imply any further restriction, so we might wonder why we ended up with (2.7.5). Note that, even starting from a general $\mathrm{GL}(4, \mathbb{R})$-invariant vector field Ξ (projecting on ξ), i.e.

$$\pounds_\Xi(\kappa^A \kappa^B \varepsilon^{A'B'}) = \xi^c \nabla_c(\kappa^A \kappa^B \varepsilon^{A'B'}) - \kappa^D \kappa^B \varepsilon^{D'B'} \check{\Xi}^a{}_d - \kappa^A \kappa^D \varepsilon^{A'D'} \check{\Xi}^b{}_d,$$

one finds oneself restricted to $\mathfrak{cso}(1, 3)$, i.e.

$$\check{\Xi}_{(AB)}^{(A'B')} = 0, \tag{2.7.6}$$

whereas the same argument applied in the *strictly* spinorial case [i.e. to an $\mathrm{SO}(p, q)^e$-invariant vector field] leads to the trivial identity

$$0 = 0,$$

i.e. to *no restriction whatsoever.*

The reason why (2.7.5) or, more generally, (2.7.6) appears is due to the particular vector space we are using to represent these "$\mathrm{GL}(4, \mathbb{R})$-spinors". Indeed, note that, although we can represent a $\mathrm{GL}(4, \mathbb{R})$-invariant vector field Ξ spinorially, explicitly

$$\Xi^{ab} = \Xi^{(AB)(A'B')} + \frac{1}{2}\Xi^{CC'}{}_{CC'}\varepsilon^{AB}\varepsilon^{A'B'} + \frac{1}{2}(\Xi^{(AB)C'}{}_{C'}\varepsilon^{A'B'} + \Xi^C{}_C{}^{(A'B')}\varepsilon^{AB}), \tag{2.7.7}$$

we *cannot* "move" a section of $S(M, g)$ with Ξ because the largest group that can act on a 2-dimensional complex vector space is $\mathrm{GL}(2, \mathbb{C})$ and

$$\dim \mathfrak{gl}(4, \mathbb{R}) \equiv 4 \cdot 4 = 16 \neq 8 = (2 \cdot 2)2 \equiv \dim_{\mathbb{R}} \mathfrak{gl}(2, \mathbb{C}),$$

unlike in the strictly spinorial case, where

$$\dim \mathfrak{so}(1, 3) \equiv \frac{4(4-1)}{2} = 6 = (2 \cdot 2 - 1)2 \equiv \dim_{\mathbb{R}} \mathfrak{sl}(2, \mathbb{C}).$$

If we nevertheless attempt to do so, we get the 9 conditions (2.7.6).

One could still wonder why we get 9 equations instead of 8. The reason is that, as we can also easily see from (2.7.7), the irreducible decomposition of $\mathfrak{gl}(4, \mathbb{R})$ is

$$\mathfrak{gl}(4, \mathbb{R}) = \mathfrak{so}(1, 3) \oplus \mathbb{R} \oplus \mathcal{V}$$

$\mathcal{V}$ being the vector space of all traceless symmetric matrices, and no combination of the dimensions of the terms on the r.h.s. adds up to 8, so that we are left with $\mathfrak{cso}(1,3) = \mathfrak{so}(1,3) \oplus \mathbb{R}$. This also explains why it is only possible to determine the real part of λ (corresponding to the term $\check{\Xi}^C{}_C$ of the general case) in (2.7.3), as already observed by Penrose & Rindler (1986), p. 102.

2.7.2 Gauge-covariant Lie derivative of G-tensor valued q-forms

Let $P(M,G)$ be a G-structure. The (connection-dependent) ***gauge-covariant Lie derivative*** $\mathrm{L}_\xi \alpha$ of a G-tensor valued q-form α on P with respect to a vector field ξ on M is defined to be (Mason & Frauendiener 1990; Woodhouse 1991; Hehl *et al.* 1995)

$$\mathrm{L}_\xi \alpha := \xi \lrcorner \mathcal{D}\alpha + \mathcal{D}(\xi \lrcorner \alpha), \tag{2.7.8}$$

where '$\mathcal{D}$' denotes the covariant exterior derivative operator associated with a G-connection ω on P. Locally,

$$\begin{aligned}\mathcal{D}\alpha^{a_1\dots a_r}_{b_1\dots b_s} := \mathrm{d}\alpha^{a_1\dots a_r}_{b_1\dots b_s} + \omega^{a_1}{}_c \wedge \alpha^{c\dots a_r}_{b_1\dots b_s} + \dots + \omega^{a_r}{}_c \wedge \alpha^{a_1\dots c}_{b_1\dots b_s} \\ - \omega^c{}_{b_1} \wedge \alpha^{a_1\dots a_r}_{c\dots b_s} - \dots - \omega^c{}_{b_s} \wedge \alpha^{a_1\dots a_r}_{b_1\dots c}\end{aligned} \tag{2.7.9}$$

[*cf.* (1.5.8)], where we set

$$\alpha^{a_1\dots a_r}_{b_1\dots b_s} := \frac{1}{q!}\,\alpha^{a_1\dots a_r}_{b_1\dots b_s \nu_1\dots\nu_q}\,\mathrm{d}x^{\nu_1}\wedge\dots\wedge\mathrm{d}x^{\nu_q}, \tag{2.7.10}$$

$\alpha^{a_1\dots a_r}_{b_1\dots b_s \nu_1\dots\nu_q} = \alpha^{a_1\dots a_r}_{b_1\dots b_s [\nu_1\dots\nu_q]}$ denoting the components of α in some suitable chart.

On recalling the definition of a gauge-natural Lie derivative and that of the horizontal lift $\hat{\xi}$ of a vector field on M onto P (*cf.* §1.5), it is now easy to see that

$$\mathrm{L}_\xi \alpha \equiv \pounds_{\hat{\xi}} \alpha, \tag{2.7.11}$$

i.e. *the gauge-covariant Lie derivative of a G-tensor valued q-form α with respect to a vector field $\xi \in \mathfrak{X}(M)$ is nothing but the gauge-natural Lie derivative of α with respect to the horizontal lift of ξ.* On a standard q-form α on M this Lie derivative reduces, of course, to the standard (natural, connection-independent) Lie derivative

$$\pounds_\xi \alpha \equiv \xi \lrcorner \mathrm{d}\alpha + \mathrm{d}(\xi \lrcorner \alpha), \tag{2.7.12}$$

whereas on a G-vector field η it reproduces its covariant derivative (associated with ω) with respect to ξ, i.e.

$$\pounds_\xi \eta \equiv \xi \lrcorner D\eta \equiv \nabla_\xi \eta,$$

consistently with (1.5.7) and (2.1.8).

2.7.3 Hehl *et al.*'s (1995) "ordinary" Lie derivative

Hehl *et al.* (1995) define the "ordinary" Lie derivative of a $\mathrm{GL}(m,\mathbb{R})$-tensor valued

q-form α with respect to a vector field ξ on M as

$$\tilde{\mathrm{L}}_\xi \alpha^{a_1 \dots a_r}_{b_1 \dots b_s} := \mathrm{L}_\xi \alpha^{a_1 \dots a_r}_{b_1 \dots b_s} - \tilde{\nabla}_c \xi^{a_1} \alpha^{c \dots a_r}_{b_1 \dots b_s} - \dots - \tilde{\nabla}_c \xi^{a_r} \alpha^{a_1 \dots c}_{b_1 \dots b_s} \\ + \tilde{\nabla}_{b_1} \xi^c \alpha^{a_1 \dots a_r}_{c \dots b_s} + \dots + \tilde{\nabla}_{b_s} \xi^c \alpha^{a_1 \dots a_r}_{b_1 \dots c},$$

where we used the abbreviated notation (2.7.10), $\alpha^{a_1 \dots a_r}_{b_1 \dots b_s \nu_1 \dots \nu_q} = \alpha^{a_1 \dots a_r}_{b_1 \dots b_s [\nu_1 \dots \nu_q]}$ denoting the components of α in some suitable chart. On using (2.7.11) and recalling (2.6.18), we readily find that

$$\tilde{\mathrm{L}}_\xi \alpha \equiv \pounds_{L\xi} \alpha,$$

which is nothing but the gauge-natural Lie derivative of α with respect to the natural lift of ξ. Of course, $\pounds_{L\xi}$ is well-defined on *any* $\mathrm{GL}(m, \mathbb{R})$-tensor, and therefore we can effortlessly extend the domain of $\tilde{\mathrm{L}}_\xi$ simply by setting

$$\tilde{\mathrm{L}}_\xi := \pounds_{L\xi}. \tag{2.7.13}$$

This Lie derivative is used quite often in physics nowadays. In principle, there is nothing wrong with this, but it must be noted that it is just a *gauge*-natural Lie derivative with respect to a *particular* lift. Therefore, *one should not claim that results found on making use of* (2.7.13) *are generally valid*.

Chapter 3

Gauge-natural field theories

> Je me suis proposé de réduire la théorie de cette Science, et l'art de résoudre les problèmes qui s'y rapportent, à des formules générales, dont le simple développement donne toutes les équations nécessaires pour la solution de chaque problème.
>
> J.-L. Lagrange, *Mécanique Analytique*, Avertissement

This chapter can be regarded as an introduction to the geometric formulation of the calculus of variations and the theory of conserved quantities in a gauge-natural context (see also Fatibene 1999; Fatibene & Francaviglia 2001; Fatibene *et al.* 2001).

In this connection, it is probably worth stressing that the reformulation of well-known classical results, such as Noether's (1918) theorems, in modern geometrical terms is not just a nice exercise of differential geometry, but provides one with tools that have proved to be extremely powerful in tackling problems which had remained unsolved until very recently (*cf.*, e.g., Francaviglia 1990; Giachetta *et al.* 1997).

3.1 Variational principle

3.1.1 Classical approach

In this section we shall describe the classical approach to the Lagrangian formulation of a field theory. We shall follow mainly Wald (1984).

We have a *field theory* whenever a (physical) system can be described in terms of two sets of variables: dependent (*field variables* or simply *fields*) and independent. The behaviour of the system is known whenever the values of the fields as functions of the independent variables are known. The fields are connected to the independent variables through one or more equations (the *field equations* or *equations of motion*), which specify their dynamics.

In this sense, classical mechanics is a field theory: time is the independent variable, the coordinates defining the system configuration are the fields, and the Euler-Lagrange equations are the field equations. Also electromagnetism is a field theory: space-time coordinates are the independent variables, the six components of the electric and magnetic fields are the field variables, and the Maxwell equations are the field equations.

In physics, the field equations are generally deduced from a *variational principle.* Consider a field theory involving an otherwise unspecified field $\psi \equiv \psi(x)$ over an m-dimensional manifold M. Usually, M is taken to be a 4-dimensional Lorentzian manifold, but this is of no particular interest here. Let $\mathscr{A}[\psi]$ be a functional of ψ, i.e. a map from the field configurations on M into the (real or complex) numbers, and let $(\psi^{\mathfrak{a}})$ denote the components of ψ with respect to some suitable basis. Let $\{\psi_\lambda\}$ be a smooth one-parameter family of field configurations starting from ψ_0 which satisfy appropriate boundary conditions, and let $(\psi^{\mathfrak{a}}_\lambda)$ denote the components of ψ_λ for each λ. We call

$$\delta\psi^{\mathfrak{a}} := \left.\frac{\partial\psi^{\mathfrak{a}}_\lambda}{\partial\lambda}\right|_{\lambda=0} \tag{3.1.1}$$

a *variation* of $\psi^{\mathfrak{a}}$. Suppose that $\mathrm{d}\mathscr{A}/\mathrm{d}\lambda$ at $\lambda = 0$ exists for all such one-parameter families starting from $\psi^{\mathfrak{a}}_0$. Moreover, suppose that $\mathrm{d}\mathscr{A}/\mathrm{d}\lambda$ can be written in the form

$$\frac{\mathrm{d}\mathscr{A}}{\mathrm{d}\lambda} = \int_D \chi_{\mathfrak{a}}\,\delta\psi^{\mathfrak{a}},$$

where $\chi_{\mathfrak{a}} \equiv \chi_{\mathfrak{a}}(x)$ is smooth in x, but otherwise unspecified, and D is a compact submanifold of M. Then we say that $\mathscr{A}$ is *functionally differentiable* at $\psi^{\mathfrak{a}}_0$. We call $\chi_{\mathfrak{a}}$ the *functional derivative* of $\mathscr{A}$ and denote it as

$$\chi_{\mathfrak{a}} =: \left.\frac{\delta\mathscr{A}}{\delta\psi^{\mathfrak{a}}}\right|_{\psi_0}.$$

Consider now a functional $\mathscr{A}$ of the form

$$\mathscr{A}[\psi] = \int_D L[\psi]\,\mathrm{d}s,$$

where L is a scalar density depending on the point x, the field ψ and a finite number of its derivatives, i.e.

$$L \equiv L(x, \psi^{\mathfrak{a}}, \partial_{\mu_1}\psi^{\mathfrak{a}}, \ldots, \partial_{\mu_1}\cdots\partial_{\mu_k}\psi^{\mathfrak{a}}),$$

and $\mathrm{d}s := \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^{m-1}$ is the standard volume element on M. Suppose that $\mathscr{A}$ is functionally differentiable and that the field configurations ψ which extremize $\mathscr{A}$,

$$\left.\frac{\delta\mathscr{A}}{\delta\psi^{\mathfrak{a}}}\right|_{\psi} = 0, \tag{3.1.2}$$

are precisely the ones which are solutions of the field equations for ψ. Then $\mathscr{A}$ is called an *action* and L a *Lagrangian density*.

Let us find now an explicit expression for (3.1.2) when L depends on the derivatives of the field only up to the *first order*, and $\delta\psi^{\mathfrak{a}} = 0$ on the boundary ∂D of D. In such a case we have:

$$\begin{aligned}\delta\mathscr{A} := \left.\frac{\mathrm{d}\mathscr{A}}{\mathrm{d}\lambda}\right|_{\lambda=0} &= \frac{\mathrm{d}}{\mathrm{d}\lambda}\int_D L(x, \psi^{\mathfrak{a}}, \partial_\mu\psi^{\mathfrak{a}})\,\mathrm{d}s \\ &= \int_D \left(\frac{\partial L}{\partial\psi^{\mathfrak{a}}}\delta\psi^{\mathfrak{a}} + \frac{\partial L}{\partial\partial_\mu\psi^{\mathfrak{a}}}\delta\partial_\mu\psi^{\mathfrak{a}}\right)\mathrm{d}s,\end{aligned} \tag{3.1.3}$$

where

$$\delta\partial_\mu\psi^{\mathfrak{a}} := \left.\frac{\partial\partial_\mu\psi^{\mathfrak{a}}_\lambda}{\partial\lambda}\right|_{\lambda=0} = \partial_\mu \left.\frac{\partial\psi^{\mathfrak{a}}_\lambda}{\partial\lambda}\right|_{\lambda=0} \equiv \partial_\mu\delta\psi^{\mathfrak{a}}. \tag{3.1.4}$$

On substituting this expression into (3.1.3) and integrating by parts, we obtain

$$\begin{aligned}\delta\mathscr{A} &= \int_D \left(\frac{\partial L}{\partial\psi^{\mathfrak{a}}} - \partial_\mu\frac{\partial L}{\partial\partial_\mu\psi^{\mathfrak{a}}}\right)\delta\psi^{\mathfrak{a}}\,\mathrm{d}s + \int_D \partial_\mu\left(\frac{\partial L}{\partial\partial_\mu\psi^{\mathfrak{a}}}\delta\psi^{\mathfrak{a}}\right)\mathrm{d}s \\ &= \int_D \left(\frac{\partial L}{\partial\psi^{\mathfrak{a}}} - \partial_\mu\frac{\partial L}{\partial\partial_\mu\psi^{\mathfrak{a}}}\right)\delta\psi^{\mathfrak{a}}\,\mathrm{d}s + \int_{\partial D} \frac{\partial L}{\partial\partial_\mu\psi^{\mathfrak{a}}}\delta\psi^{\mathfrak{a}}\,\mathrm{d}s_\mu,\end{aligned} \tag{3.1.5}$$

where in the second line we used Stokes's theorem and $\mathrm{d}s_\mu := \partial_\mu \lrcorner\, \mathrm{d}s$ (*cf.* §1.1). Now, the second integral in (3.1.5) vanishes since $\delta\psi^{\mathfrak{a}} = 0$ on ∂D. Therefore, condition (3.1.2) reduces to the vanishing of the first integral in (3.1.5) or, equivalently,

$$\frac{\partial L}{\partial\psi^{\mathfrak{a}}} - \partial_\mu\frac{\partial L}{\partial\partial_\mu\psi^{\mathfrak{a}}} = 0, \tag{3.1.6}$$

since both D and $\delta\psi^{\mathfrak{a}}$ are arbitrary. Eqs. (3.1.6) are called the *Euler-Lagrange equations*.

Remark 3.1.1. At this stage, the geometric framework is not quite clear. Moreover, observe a few notational inconsistencies, e.g. in (3.1.6). In the denominator of the second term on the l.h.s, $\partial_\mu\psi^{\mathfrak{a}}$ has a "formal" meaning, i.e. it is the "variable" with respect to which we are differentiating the Lagrangian density and *must not* be regarded as the partial derivative of $\psi^{\mathfrak{a}}(x)$. Furthermore, the operator ∂_μ in the numerator is ambiguous too: in principle, it could be regarded as either a partial derivative with respect to x^μ or a total derivative to which the chain rule applies. The latter is the meaning it actually has, but *only once* $\partial L/\partial\partial_\mu\psi^{\mathfrak{a}}$ is evaluated on a solution of the field equations, otherwise $\psi^{\mathfrak{a}}$ and x are to be regarded as *independent variables*. Indeed, recall classical mechanics, where the Euler-Lagrange equations read

$$\frac{\partial L}{\partial q^i} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial u^i} = 0, \tag{3.1.7}$$

and $u^i = \mathrm{d}q^i/\mathrm{d}t$ *only* on the equations of motion, and the time derivative appearing in (3.1.7) is, with reason, a total one.

3.1.2 Geometric formulation on gauge-natural bundles

A field theory can be phrased in geometrical terms by means of a fibred manifold (B, M, π), the base M representing the space of independent variables, the total space B representing the space of fields, and the projection π associating with each point $x \in M$ (independent variable) the set $B_x := \pi^{-1}(x)$ of all possible values of the fields at x. The system behaviour is given by a (usually only local) section σ of (B, M, π). Such a section is the geometric equivalent of the field ψ of the previous section. Therefore, the space of all possible fields is the space of all local sections of (B, M, π). We call the sections that are solutions of the field equations *critical*, whereas (B, M, π) is known as the ***configuration bundle*** of the theory.

Now, although the calculus of variations could be entirely developed on fibred manifolds (*cf.*, e.g., Giachetta *et al.* 1997), for our present purposes these are far too general objects. Therefore, in the sequel the configuration bundle of the theory will be assumed to be a gauge-natural bundle P_λ associated with some principal bundle $P(M, G)$ (*cf.* §1.10). P is then called the ***structure bundle*** (of the theory). Indeed, gauge-natural bundles form a category large enough to encompass all known classical field theories (*cf.* Eck 1981; Kolář *et al.* 1993; Fatibene 1999). At the same time, this specialization to gauge-natural bundles will enable us to give a more concrete characterization of the physical systems under consideration.

The basic ingredient of a *geometric* (*Lagrangian*) *field theory* is its ***Lagrangian***, i.e. a base-preserving morphism

$$\mathfrak{L}\colon J^k P_\lambda \to \textstyle\bigwedge^m T^*M, \tag{3.1.8}$$

$\bigwedge^m T^*M$ being the vector bundle of m-forms on M and m the dimension of M, or, equivalently, a horizontal m-form $\mathfrak{L} \in \Omega^m_0(J^k P_\lambda)$ (*cf.* §1.8). The "k-th order jet bundle" $J^k P_\lambda$, whose precise definition was given in §1.7, can be thought of simply as the space of the field variables together with all their derivatives up to the k-th order. We stress that, in this geometric approach, the fields and their derivatives (together with the independent variables, of course) are to be regarded as *coordinates* on $J^k P_\lambda$. Indeed, one of the main advantages of this formalism is that jets of fibre bundles (when sections are identified by a finite number of terms of their Taylor series) form smooth *finite-dimensional* manifolds. Therefore, the dynamics of field systems is defined on a finite-dimensional configuration space.

The integer k is called the *order of the Lagrangian*. In the rest of this thesis, we shall be mainly concerned with first order Lagrangians; therefore, as a rule, in the sequel we shall only discuss the first order case in some detail, limiting ourselves to simply state the corresponding results for higher order Lagrangians. Indeed, most physical theories are described by first order Lagrangians, one remarkable exception being general relativity in the Einstein-Hilbert approach (*cf.* §§3.3.5 and 4.5). A geometric field theory based on a k-th order Lagrangian is called a *k-th order field theory*.

The variation of the field (3.1.1) is geometrically represented by a vertical[1] vector field Υ on $J^1 P_\lambda$ (*cf.* §1.2). The variation of a (first order) Lagrangian $\mathfrak{L}$ is then simply defined as the Lie derivative of $\mathfrak{L}$ with respect to the pair of vector fields $(J^1\Upsilon, 0)$ in the sense of formula (2.1.3), where $J^1\Upsilon$ denotes the first oder jet prolongation of Υ, i.e. locally

$$J^1\Upsilon = \Upsilon^{\mathfrak{a}} \frac{\partial}{\partial y^{\mathfrak{a}}} + \mathrm{d}_\mu \Upsilon^{\mathfrak{a}} \frac{\partial}{\partial y^{\mathfrak{a}}{}_\mu}, \tag{3.1.9}$$

$(\Upsilon^{\mathfrak{a}})$ denoting the components of Υ [*cf.* (1.9.3)]. Thus, we have

$$\pounds_{(J^1\Upsilon,0)}\mathfrak{L} \equiv \langle \mathrm{d}\mathfrak{L}, J^1\Upsilon \rangle = \left(\frac{\partial L}{\partial y^{\mathfrak{a}}} \Upsilon^{\mathfrak{a}} + \frac{\partial L}{\partial y^{\mathfrak{a}}{}_\mu} \mathrm{d}_\mu \Upsilon^{\mathfrak{a}} \right) \mathrm{d}s, \tag{3.1.10}$$

[1]We are just considering *variations of the field*, not of the point $x \in M$ at which it is evaluated. The "full variation" of the Lagrangian enters the discussion of conserved quantities and is given by (the second component of) (3.2.4) below.

where L is the scalar density intrinsically defined by the local decomposition

$$\mathfrak{L} = L\,\mathrm{d}s, \tag{3.1.11}$$

where $\mathrm{d}s$ is as in §1.1. Observe that the notation is consistent since

$$\pounds_{(J^1\Upsilon,0)} y^{\mathfrak{a}} \equiv \langle \mathrm{d}y^{\mathfrak{a}}, J^1\Upsilon\rangle = \frac{\partial y^{\mathfrak{a}}}{\partial y^{\mathfrak{b}}}\Upsilon^{\mathfrak{b}} = \Upsilon^{\mathfrak{a}} \equiv -\tilde{\pounds}_{\Upsilon} y^{\mathfrak{a}},$$

i.e. $\Upsilon^{\mathfrak{a}} \leftrightarrow \delta\psi^{\mathfrak{a}}$ and $\pounds_{(J^1\Upsilon,0)} \leftrightarrow \delta$ (*cf.* §3.1.1).

On applying the Leibniz rule to the second term on the r.h.s. of (3.1.10), the variation of $\mathfrak{L}$ can be rewritten as[2]

$$\langle \mathrm{d}\mathfrak{L}, J^1\Upsilon\rangle = \langle e(\mathfrak{L}), \Upsilon\rangle + \mathrm{d}_{\mathrm{H}}\langle f(\mathfrak{L}), \Upsilon\rangle, \tag{3.1.12}$$

where $e(\mathfrak{L})$ is the ***Euler-Lagrange morphism***, locally given by

$$e(\mathfrak{L}) \equiv \bar{\mathrm{d}}y^{\mathfrak{a}} \otimes e_{\mathfrak{a}}(\mathfrak{L}) := \left(\frac{\partial L}{\partial y^{\mathfrak{a}}} - \mathrm{d}_\mu \frac{\partial L}{\partial y^{\mathfrak{a}}{}_\mu}\right) \bar{\mathrm{d}}y^{\mathfrak{a}} \otimes \mathrm{d}s, \tag{3.1.13}$$

and

$$f(\mathfrak{L}) \equiv f_{\mathfrak{a}}{}^\mu\, \bar{\mathrm{d}}y^{\mathfrak{a}} \otimes \mathrm{d}s_\mu := \frac{\partial L}{\partial y^{\mathfrak{a}}{}_\mu}\, \bar{\mathrm{d}}y^{\mathfrak{a}} \otimes \mathrm{d}s_\mu, \tag{3.1.14}$$

$\{\bar{\mathrm{d}}y^{\mathfrak{a}}\}$ denoting the fibre basis of $V^*P_\lambda := (VP_\lambda)^*$ defined by requiring $\langle \bar{\mathrm{d}}y^{\mathfrak{a}}, \partial_{\mathfrak{b}}\rangle = \delta^{\mathfrak{a}}{}_{\mathfrak{b}}$. Identity (3.1.12) is called the ***first variation formula*** and the $f_{\mathfrak{a}}{}^\mu$'s the (first order) ***momenta***. This calculation can be generalized (in a straightforward but rather technical way) to the k-th order case. Indeed, we have the following

Proposition 3.1.2. *Let $\mathfrak{L}$ be a k-th order Lagrangian on a gauge-natural bundle P_λ, $k \geqslant 1$. There exist a* global *morphism $f(\mathfrak{L},\Gamma)\colon J^{2k-1}P_\lambda \to V^*J^{k-1}P_\lambda \otimes \bigwedge^{m-1}T^*M$ and a* unique global *morphism $e(\mathfrak{L})\colon J^{2k}P_\lambda \to V^*P_\lambda \otimes \bigwedge^m T^*M$ such that*

$$(\pi^{2k}_k)^*\langle \mathrm{d}\mathfrak{L}, J^k\Upsilon\rangle = \langle e(\mathfrak{L}), \Upsilon\rangle + \mathrm{d}_{\mathrm{H}}\langle f(\mathfrak{L},\Gamma), J^{k-1}\Upsilon\rangle \tag{3.1.15}$$

for any vertical vector field Υ on P_λ. Locally,

$$e(\mathfrak{L}) = \left(\frac{\partial L}{\partial y^{\mathfrak{a}}} + \sum_{1\leqslant|\boldsymbol{\mu}|\leqslant k} (-1)^{|\boldsymbol{\mu}|}\, \mathrm{d}_{\boldsymbol{\mu}} \frac{\partial L}{\partial y^{\mathfrak{a}}{}_{\boldsymbol{\mu}}}\right) \bar{\mathrm{d}}y^{\mathfrak{a}} \otimes \mathrm{d}s.$$

Proof. See, e.g., Kolář *et al.* (1993), §49.3. □

In the sequel, we shall use the classical notation $\delta\mathfrak{L}$ for $(\pi^{2k}_k)^*\langle \mathrm{d}\mathfrak{L}, J^k\Upsilon\rangle \equiv (\pi^{2k}_k)^*\pounds_{(J^k\Upsilon,0)}\mathfrak{L}$ whenever no confusion can arise. Unlike $e(\mathfrak{L})$, $f(\mathfrak{L},\Gamma)$ is not uniquely determined and depends in general on a linear connection Γ on M. In the case $k = 1, 2$, though, this

[2]More precisely, the l.h.s. of (3.1.12) should read $(\pi^2_1)^*\langle \mathrm{d}\mathfrak{L}, J^1\Upsilon\rangle$ [*cf.* §1.7 and (3.1.15) below].

dependence disappears[3] and $f(\mathfrak{L},\Gamma)=f(\mathfrak{L})$ locally reads

$$f(\mathfrak{L})=\left[\left(\frac{\partial L}{\partial y^{\mathfrak{a}}{}_{\mu}}-(k-1)\mathrm{d}_{\nu}\frac{\partial L}{\partial y^{\mathfrak{a}}{}_{\mu\nu}}\right)\bar{\mathrm{d}}y^{\mathfrak{a}}+(k-1)\frac{\partial L}{\partial y^{\mathfrak{a}}{}_{\mu\nu}}\bar{\mathrm{d}}y^{\mathfrak{a}}{}_{\nu}\right]\otimes\mathrm{d}s_{\mu},$$

where $\bar{\mathrm{d}}y^{\mathfrak{a}}{}_{\nu}$ is defined by requiring $\langle\bar{\mathrm{d}}y^{\mathfrak{a}}{}_{\nu},\partial_{\mathfrak{b}}{}^{\rho}\rangle=\delta^{\mathfrak{a}}{}_{\mathfrak{b}}\delta^{\rho}{}_{\nu}$ [compare with (3.1.14) above]. The global morphism $f(\mathfrak{L},\Gamma)$ is called the ***Poincaré-Cartan morphism***.

Let us now come back to the first order case. On following now the same line of argument that led from (3.1.5) to (3.1.6), the field equations are found to be

$$e(\mathfrak{L})\circ j^2\sigma=0 \tag{3.1.16}$$

for any critical section $\sigma\colon M\to P_\lambda$, or, locally,

$$\left(\frac{\partial L}{\partial y^{\mathfrak{a}}}-\mathrm{d}_{\mu}\frac{\partial L}{\partial y^{\mathfrak{a}}{}_{\mu}}\right)\circ j^2\sigma=0, \tag{3.1.16$'$}$$

by virtue of (3.1.13) and consistently with (3.1.6). The field equations are second-order: indeed, the second term in brackets on l.h.s. of (3.1.16′) depends in general on the fields, and their first and second derivatives. On the other hand, the momenta depend in general on the fields and their first derivatives.

The term appearing as a horizontal differential in (3.1.15) [or (3.1.12)] is called the “boundary term”, since in the formula of the variation of the action it goes over to an integral on the boundary of the domain D. If two Lagrangians differ by a boundary term, then the corresponding field equations turn out to be the same, in which case the Lagrangians are said to be (***dynamically***) ***equivalent***. Indeed, let

$$\mathfrak{L}':=\mathfrak{L}+\mathrm{d}_{\mathrm{H}}\beta,$$

where $\mathfrak{L}$ is a k-th order Lagrangian on P_λ and β is a horizontal $(m-1)$-form on J^kP_λ such that $\mathrm{d}_{\mathrm{H}}\beta\in\Omega^m_0(J^kP_\lambda)$. Then, $\mathfrak{L}'$ is also a k-th order Lagrangian on P_λ. Moreover,

$$\begin{aligned}\delta\mathfrak{L}'&=\delta\mathfrak{L}+\delta\mathrm{d}_{\mathrm{H}}\beta\\&=\langle e(\mathfrak{L}),\Upsilon\rangle+\mathrm{d}_{\mathrm{H}}\langle f(\mathfrak{L},\Gamma),J^{k-1}\Upsilon\rangle+\delta\mathrm{d}_{\mathrm{H}}\beta,\end{aligned}$$

where we used (3.1.15) and set $\delta:=\pounds_{(J^k\Upsilon,0)}$. However, the operators δ and d_{H} commute. This can be seen in various ways: from the identification of δ with the variation defined in §3.1.1, and hence from (3.1.4); in local coordinates, from expression (3.1.10); on applying the *covariant* functor J^1 to (2.1.3). Therefore,

$$\delta\mathfrak{L}'=\langle e(\mathfrak{L}),\Upsilon\rangle+\mathrm{d}_{\mathrm{H}}\Big(\langle f(\mathfrak{L},\Gamma),J^{k-1}\Upsilon\rangle+\delta\beta\Big). \tag{3.1.17}$$

However, by virtue of Proposition 3.1.2, for each Lagrangian $\mathfrak{L}'$, there is a *unique* Euler-

[3]More precisely, $f(\mathfrak{L},\Gamma)$ is uniquely determined for $m=1$ (any k) and $k=1$ (any m). For $m\geqslant 2$ and $k\geqslant 2$ uniqueness is lost, but for $k=2$ (any m) there still is a *unique canonical* choice.

Lagrange morphism $e(\mathfrak{L}')$ such that a decomposition like (3.1.17) holds. Hence,

$$e(\mathfrak{L}') \equiv e(\mathfrak{L}),$$

as claimed. □

Proving the converse, i.e. that all Lagrangians equivalent to a given one differ from one another by a boundary term, represents one of the most difficult problems of the geometric calculus of variations and, in the k-th order case, was only recently solved by Krupka & Musilová (1998). In the present, gauge-natural, context, their result can be rephrased as follows.

Theorem 3.1.3. *Two k-th order Lagrangians $\mathfrak{L}$ and $\mathfrak{L}'$ on a gauge-natural bundle P_λ are equivalent iff, locally, they differ from each other by the horizontal part of the exterior differential of an $(m-1)$-form χ on $J^{k-1}P_\lambda$.*

In the *first* order case, this result admits a further refinement, whereby two first order Lagrangians on P_λ are equivalent iff they differ from each other by the horizontal part of a (global) *closed m-form* on P_λ (Krupka 1974).

3.2 Symmetries and conserved quantities

Roughly speaking, a "symmetry" of a physical system is a correspondence associating with any possible history of the system another such history of the same system. More precisely, we can give the following

Definition 3.2.1. Let P_λ be a gauge-natural bundle associated with some principal bundle $P(M,G)$ over an m-dimensional manifold M, as in the previous section. A *configuration bundle automorphism* (φ, Φ), i.e. an automorphism Φ of P_λ over a diffeomorphism $\varphi\colon M \to M$, is called a ***symmetry*** of a geometric field theory described by a k-th order Lagrangian $\mathfrak{L}\colon J^kP_\lambda \to \bigwedge^m T^*M$ if

$$e(\mathfrak{L}) \circ j^{2k}(\Phi \circ \sigma \circ \varphi^{-1}) = 0$$

for any critical section $\sigma\colon M \to P_\lambda$.

Now, since $\mathfrak{L}\colon J^kP_\lambda \to \bigwedge^m T^*M$ and $e(\mathfrak{L})\colon J^{2k}P_\lambda \to V^*P_\lambda \otimes \bigwedge^m T^*M$, for *any* configuration bundle automorphism (φ, Φ) we must have[4]

$$e(\textstyle\bigwedge^m T^*\varphi^{-1} \circ \mathfrak{L} \circ J^k\Phi) = (V^*\Phi^{-1} \otimes \textstyle\bigwedge^m T^*\varphi^{-1}) \circ e(\mathfrak{L}) \circ J^{2k}\Phi, \tag{3.2.1}$$

where $V^*\Phi := T^*\Phi|_{V^*P_\lambda}$.

Definition 3.2.2. A configuration bundle automorphism (φ, Φ) is called a ***generalized invariant transformation*** if it leaves the Euler-Lagrange morphism associated with a k-th order Lagrangian $\mathfrak{L}$ on P_λ unchanged, i.e. if $(V^*\Phi^{-1} \otimes \bigwedge^m T^*\varphi^{-1}) \circ e(\mathfrak{L}) \circ J^{2k}\Phi = e(\mathfrak{L})$.

[4] Remember the "direction" of our cotangent maps as defined in §1.1.

In the sequel we shall deal with a more restricted class of symmetries, which is nonetheless large enough to encompass most cases of physical interest.

Definition 3.2.3. A configuration bundle automorphism (φ, Φ) is called an ***invariant transformation*** (or a ***Lagrangian symmetry***) for a k-th order Lagrangian $\mathfrak{L}\colon J^kP_\lambda \to \bigwedge^m T^*M$ if it leaves $\mathfrak{L}$ unchanged, i.e. if $\bigwedge^m T^*\varphi^{-1} \circ \mathfrak{L} \circ J^k\Phi = \mathfrak{L}$.

It follows immediately from (3.2.1) and Definition 3.2.3 that any invariant transformation is a generalized invariant transformation. In its turn, any generalized invariant transformation is a symmetry. Indeed, from (1.7.2) and Definition 3.2.2 it follows that

$$\begin{aligned} e(\mathfrak{L}) \circ j^{2k}(\Phi \circ \sigma \circ \varphi^{-1}) &= e(\mathfrak{L}) \circ J^{2k}\Phi \circ j^{2k}\sigma \circ \varphi^{-1} \\ &= (V^*\Phi \otimes {\textstyle\bigwedge^m} T^*\varphi) \circ e(\mathfrak{L}) \circ j^{2k}\sigma \circ \varphi^{-1}, \end{aligned}$$

which clearly vanishes if $e(\mathfrak{L}) \circ j^{2k}\sigma = 0$. □

Now, the infinitesimal version of Definition 3.2.3 is the following.

Definition 3.2.4. A vector field Ξ generating a one-parameter group $\{\Phi_t\}$ of invariant transformations is called an ***infinitesimal invariant transformation*** (or an ***infinitesimal Lagrangian symmetry***).

Definition 3.2.5. Let $\mathrm{Aut}(P_\lambda)$ denote the group of all *induced* automorphisms of P_λ [*cf.* (1.10.4)]. We shall say that a k-th order Lagrangian on P_λ is $\mathrm{Aut}(P_\lambda)$***-invariant*** if any induced automorphism of P_λ is a Lagrangian symmetry (and any induced vector field on P_λ is an infinitesimal Lagrangian symmetry).

We are now in a position to refine our definition of a geometric field theory on a gauge-natural bundle P_λ.

Definition 3.2.6. A ***k-th order gauge-natural (Lagrangian) field theory*** is a geometric field theory on a gauge-natural bundle P_λ in which the fields are represented by (local) sections of P_λ and the equations of motion can be formally written as

$$e(\mathfrak{L}) \circ j^{2k}\sigma = 0 \tag{3.2.2}$$

for some suitable $\mathrm{Aut}(P_\lambda)$-invariant k-th order Lagrangian $\mathfrak{L}$ on P_λ and some (critical) section $\sigma\colon M \to P_\lambda$. Whenever an identity holds only modulo equation (3.2.2), we shall call it a *weak identity* (as opposed to *strong*), or say that it holds "on shell", and use the symbol '$\approx$' instead of the equals sign. In particular, we shall write equation (3.2.2) itself simply as $e(\mathfrak{L}) \approx 0$.

All known classical Lagrangian field theories such as all standard gravitational field theories (including, in particular, Einstein's general relativity and the Einstein-Cartan theory), electromagnetism, the Yang-Mills theory, bosonic and fermionic matter field theories, topological field theories—as well as all their possible mutual couplings—are Lagrangian field theories on some suitable gauge-natural (vector or affine) bundle (*cf.* Eck 1981; Kolář *et al.* 1993; Fatibene 1999).

Remark 3.2.7. Note that, if one starts from Definition 3.2.6, there is no need to introduce the concepts of an action or a variation, and one can work completely within the boundaries of a finite-dimensional formalism.

Proposition 3.2.8. *Let Ξ_λ be a vector field on P_λ induced by a G-invariant vector field Ξ on P projecting on a vector field ξ on M, and $\mathfrak{L}$ an* $\mathrm{Aut}(P_\lambda)$*-invariant k-th order Lagrangian on P_λ. Then,*

$$(\pi_k^{2k})^* \langle \mathrm{d}\mathfrak{L}, J^k \tilde{\mathcal{L}}_\Xi y \rangle = \mathrm{d_H}(\xi \rfloor \mathfrak{L}). \tag{3.2.3}$$

Proof. The result readily follows from Definition 3.2.4 and the properties of the formal generalized Lie derivative, taking into account the isomorphism $J^kVP_\lambda \cong VJ^kP_\lambda$ locally given by the identification $(x^\lambda, y^\mathfrak{a}, \dot{y}^\mathfrak{b}, y^\mathfrak{c}{}_\mu, (\dot{y}^\mathfrak{d})_\nu) \cong (x^\lambda, y^\mathfrak{a}, y^\mathfrak{c}{}_\mu, \dot{y}^\mathfrak{b}, (y^\mathfrak{d}{}_\nu)^{\cdot} := (\dot{y}^\mathfrak{d})_\nu)$. In the *first order* case, one explicitly proceeds as follows. By Definition 3.2.3 a one-parameter group of (induced) Lagrangian symmetries $\{\varphi_t, (\Phi_t)_\lambda\}$ satisfies

$$\mathfrak{L} \circ J^k(\Phi_t)_\lambda = \textstyle\bigwedge^m T^*\varphi_t \circ \mathfrak{L}$$

On differentiating this expression with respect to t at $t = 0$, one gets

$$\tilde{\mathcal{L}}_{(J^k\Xi_\lambda, \wedge^m T^*\xi)}\mathfrak{L} \equiv T\mathfrak{L} \circ J^k\Xi_\lambda - \textstyle\bigwedge^m T^*\xi \circ \mathfrak{L} = 0, \tag{3.2.4}$$

Ξ_λ being the infinitesimal counterpart of $\{(\Phi_t)_\lambda\}$ and ξ its projection on M. Specializing now to the case $k = 1$, (the second component of) formula (3.2.4) can be locally rewritten as

$$\begin{aligned} 0 &= \langle \mathrm{d}L, J^1\Xi_\lambda \rangle + \partial_\mu \xi^\mu L \\ &= \xi^\mu \partial_\mu L + \Xi^\mathfrak{a} \partial_\mathfrak{a} L + (\mathrm{d}_\mu \Xi^\mathfrak{a} - y^\mathfrak{a}{}_\nu \partial_\mu \xi^\nu) \partial_\mathfrak{a}{}^\mu L + \partial_\mu \xi^\mu L, \end{aligned} \tag{3.2.5}$$

where we used (1.9.3). Consider now the expressions for the formal generalized Lie derivative of $y^\mathfrak{a}$ and $y^\mathfrak{a}{}_\mu$, given by (2.1.11$'$) and (2.1.13), respectively. On inserting these two expressions into (3.2.5) and recalling that [*cf.* (1.8.11)]

$$\mathrm{d}_\mu(L\xi^\mu) = \partial_\mu \xi^\mu L + \xi^\mu \partial_\mu L + \xi^\mu y^\mathfrak{a}{}_\mu \partial_\mathfrak{a} L + \xi^\mu y^\mathfrak{a}{}_{\mu\nu} \partial_\mathfrak{a}{}^\nu L,$$

we get immediately

$$\mathrm{d}_\mu(L\xi^\mu) = \partial_\mathfrak{a} L \, \tilde{\mathcal{L}}_\Xi y^\mathfrak{a} + \partial_\mathfrak{a}{}^\mu L \, \tilde{\mathcal{L}}_\Xi y^\mathfrak{a}{}_\mu, \tag{3.2.6}$$

which is nothing but the coordinate expression of (3.2.3) in the first order case. □

Identity (3.2.3) is known as the ***fundamental identity***. Combining (3.1.12) and (3.2.3) we get

$$\mathrm{d_H} E(\mathfrak{L}, \Xi) = W(\mathfrak{L}, \Xi), \tag{3.2.7}$$

where we set

$$E(\mathfrak{L}, \Xi) := -\xi \rfloor \mathfrak{L} + \langle f(\mathfrak{L}, \Gamma), J^{k-1}\tilde{\mathcal{L}}_\Xi y \rangle \tag{3.2.8}$$

and

$$W(\mathfrak{L}, \Xi) := -\langle e(\mathfrak{L}), \tilde{\mathcal{L}}_\Xi y \rangle. \tag{3.2.9}$$

$E(\mathfrak{L}, \Xi)$ is called the ***Noether current*** and $W(\mathfrak{L}, \Xi)$ the ***work form***. For a first order theory, formula (3.2.8) explicitly reads

$$E(\mathfrak{L}, \Xi) = (-L\xi^\mu + f_\mathfrak{a}{}^\mu \tilde{\mathcal{L}}_\Xi y^\mathfrak{a})\, \mathrm{d}s_\mu, \tag{3.2.10}$$

where we are assuming (3.1.11), as usual. Formula (3.2.7) is the generalization of the (***first***) ***Noether* (1918) *theorem*** to the geometric framework of jet prolongations of gauge-natural bundles. Indeed, if we define

$$\begin{aligned} E(\mathfrak{L}, \Xi, \sigma) &:= (j^{2k-1}\sigma)^* E(\mathfrak{L}, \Xi), \\ W(\mathfrak{L}, \Xi, \sigma) &:= (j^{2k}\sigma)^* W(\mathfrak{L}, \Xi), \end{aligned}$$

we have

$$\mathrm{d}E(\mathfrak{L}, \Xi, \sigma) = W(\mathfrak{L}, \Xi, \sigma)$$

and, whenever σ is a critical section,

$$\mathrm{d}E(\mathfrak{L}, \Xi, \sigma) = 0. \tag{3.2.11}$$

Thus, given an infinitesimal Lagrangian symmetry Ξ_λ, we have a whole class of currents $E(\mathfrak{L}, \Xi, \sigma)$ (one for each solution σ), which are closed $(m-1)$-forms on M. We stress that the Noether current $E(\mathfrak{L}, \Xi)$ is defined at the bundle level and is *canonically* associated with the Lagrangian $\mathfrak{L}$. It is only at a *later* stage that it is evaluated on a section $\sigma\colon M \to P_\lambda$, thereby giving $E(\mathfrak{L}, \Xi, \sigma)$.

Remark 3.2.9. Note that Noether theorem is an *identity* [a strong one in the sense of (3.2.7) or a weak one in sense of (3.2.11)] ensuing *solely* from the $\mathrm{Aut}(P_\lambda)$-invariance of the Lagrangian. In the classical terminology, $E(\mathfrak{L}, \Xi)$ is known as a *first integral* (*of the motion*).

Since $E(\mathfrak{L}, \Xi, \sigma)$ is an $(m-1)$-form on M, it can be integrated over an $(m-1)$-dimensional region Σ, namely a compact submanifold $\Sigma \hookrightarrow M$ with boundary $\partial\Sigma$.

Definition 3.2.10. The real functional

$$Q_\Sigma(\mathfrak{L}, \Xi, \sigma) := \int_\Sigma E(\mathfrak{L}, \Xi, \sigma) \tag{3.2.12}$$

is called the ***conserved quantity*** (or ***charge***) along σ with respect to Ξ and Σ.

Clearly, if σ is a critical section, and two compact $(m-1)$-submanifolds $\Sigma, \Sigma' \hookrightarrow M$ form the homological boundary ∂D of a compact m-dimensional domain $D \subseteq M$, from (3.2.12), Stokes's theorem and (3.2.11) we readily obtain

$$\begin{aligned} Q_{\Sigma'}(\mathfrak{L}, \Xi, \sigma) - Q_\Sigma(\mathfrak{L}, \Xi, \sigma) &= \textstyle\int_{\Sigma'} E(\mathfrak{L}, \Xi, \sigma) - \int_\Sigma E(\mathfrak{L}, \Xi, \sigma) \\ &= \textstyle\oint_{\partial D \equiv \Sigma' \setminus \Sigma} E(\mathfrak{L}, \Xi, \sigma) \\ &= \textstyle\int_D \mathrm{d}E(\mathfrak{L}, \Xi, \sigma) = 0. \end{aligned} \tag{3.2.13}$$

Remark 3.2.11. In classical mechanics (*cf.* §3.3.1 below), $M = \mathbb{R}$, $E(\mathfrak{L}, \Xi, \sigma)$ is a 0-form, i.e. an $\mathbb{R}$-valued function on M, and Σ, Σ' are just two points $t, t' \in \mathbb{R}$. Therefore (3.2.13) becomes

$$Q_{t'}(\mathfrak{L}, \Xi, \sigma) = Q_t(\mathfrak{L}, \Xi, \sigma),$$

i.e. $Q_t(\mathfrak{L}, \Xi, \sigma)$ is indeed a quantity conserved in time along any solution $\sigma\colon t \mapsto (t, q^i = \sigma^i(t))$ of the field equations. This motivates the name of a "conserved quantity" for $Q_\Sigma(\mathfrak{L}, \Xi, \sigma)$ in the general case.

Since $E(\mathfrak{L}, \Xi, \sigma)$ is closed on shell, in field theories where $m > 1$ we may ask ourselves whether it is also exact, i.e. whether there exists an $(m-2)$-form $U(\mathfrak{L}, \Xi, \sigma)$ on M such that

$$E(\mathfrak{L}, \Xi, \sigma) = \mathrm{d}U(\mathfrak{L}, \Xi, \sigma). \tag{3.2.14}$$

If this happens to be the case, then we can express $Q_\Sigma(\mathfrak{L}, \Xi, \sigma)$ as an $(m-2)$-dimensional integral over the boundary $\partial\Sigma$ of Σ. Indeed, on using (3.2.14) and Stokes's theorem, we have

$$\begin{aligned} Q_\Sigma(\mathfrak{L}, \Xi, \sigma) &\equiv \textstyle\int_\Sigma E(\mathfrak{L}, \Xi, \sigma) \\ &= \textstyle\oint_{\partial\Sigma} U(\mathfrak{L}, \Xi, \sigma). \end{aligned} \tag{3.2.15}$$

Actually, it is possible to prove the following fundamental

Theorem 3.2.12 (Fatibene 1999[5]). *Let $P_\ell := W^{1,1}P \times_\ell \mathcal{A}$ denote the bundle of G-connections on a principal bundle $P(M, G)$ and $L^2M_{\tilde{\ell}} := L^2M \times_{\tilde{\ell}} T^1{}_2(\mathbb{R}^m)$ the bundle of (natural) linear connections on M (cf. Examples 1.10.16 and 1.10.17). Now, let P_λ be a gauge-natural bundle associated with P and suppose there exist two base-preserving morphisms $\omega\colon J^kP_\lambda \to P_\ell$ and $\Gamma\colon J^kP_\lambda \to L^2M_{\tilde{\ell}}$ associating with each section $\sigma\colon M \to P_\lambda$ a G-connection $\omega \circ j^k\sigma$ on P and a linear connection $\Gamma \circ j^k\sigma$ on M, respectively. Then, the Noether current is exact on shell for all k-th order gauge-natural field theories on P_λ, regardless of the topology of M.*

We shall not prove Theorem 3.2.12 in the general case, but shall limit ourselves to explicitly show that it holds true for all the theories we shall deal with. We stress that this important result can only be achieved since Noether's theorem has been formulated in terms of fibred morphisms rather than directly in terms of forms on M. In particular, we shall give the following

Definition 3.2.13. If the Noether current $E(\mathfrak{L}, \Xi)$ can be written as

$$E(\mathfrak{L}, \Xi) = \tilde{E}(\mathfrak{L}, \Xi) + \mathrm{d_H}U(\mathfrak{L}, \Xi), \tag{3.2.16}$$

where $\tilde{E}(\mathfrak{L}, \Xi, \sigma) := (j^{2k-1}\sigma)^*\tilde{E}(\mathfrak{L}, \Xi)$ vanishes for any critical section σ, then we shall call $\tilde{E}(\mathfrak{L}, \Xi)$ and $U(\mathfrak{L}, \Xi)$ the ***reduced (Noether) current*** and the ***superpotential*** associated with $\mathfrak{L}$, respectively. Whenever the splitting (3.2.16) holds, then it is immediate to see that $U(\mathfrak{L}, \Xi, \sigma) := (j^{2k-1}\sigma)^*U(\mathfrak{L}, \Xi)$ satisfies equation (3.2.14) for any critical section σ.

[5] The proof closely follows the analogous one for natural field theories due to Robutti (1984) (see also Ferraris *et al.* 1986, 1987). The existence of the morphisms in question is a technical condition required for ensuring a global decomposition of the Noether current (see also §3.2.1 below). It is interesting to note, though, that in many theories of physical interest a G-connection and/or a linear connection naturally appear as dynamical variables.

We stress that Noether currents and superpotentials have an indirect physical meaning, in that they provide the corresponding conserved quantities upon integration. What is really physically meaningful, though, are the values of these integrals, which only depend on the cohomology class, not on the chosen representative. In other words, Noether currents are defined modulo exact $(m-1)$-forms, superpotentials modulo closed $(m-2)$-forms.

Finally, one might be interested in what happens to Noether currents and superpotentials (and, hence, to the conserved quantities) when the Lagrangian $\mathfrak{L}$ of the theory is replaced by an equivalent Lagrangian $\mathfrak{L}'$, which, by virtue of Definition 3.2.2, is the same as asking what the conserved quantities associated with a generalized invariant transformation $\mathfrak{L} \mapsto \mathfrak{L}'$ are. From Theorem 3.1.3 it follows that, locally, $\mathfrak{L}' = \mathfrak{L} + h(\mathrm{d}\chi)$, where χ is an $(m-1)$-form on $J^{k-1}P_\lambda$. Now, comparison between (3.1.15) and (3.2.8) tells us that we can read off the Noether current associated with $\mathfrak{L}'$ from its first variation, which will be given by (3.1.17) if we just set $\beta := h(\chi)$. Explicitly,

$$E(\mathfrak{L}', \Xi) = E(\mathfrak{L}, \Xi) + \pounds_\Xi \beta - \xi \rfloor \mathrm{d_H}\beta,$$

where we used (1.8.17). But β is in $\Omega_0^{m-1}(J^k P_\lambda)$ by definition; accordingly, $\pounds_\Xi \beta = \pounds_\xi \beta \equiv \xi \rfloor \mathrm{d_H}\beta + \mathrm{d_H}(\xi \rfloor \beta)$, whence

$$\begin{aligned} \tilde{E}(\mathfrak{L}', \Xi) &= \tilde{E}(\mathfrak{L}, \Xi), \\ U(\mathfrak{L}', \Xi) &= U(\mathfrak{L}, \Xi) + \xi \rfloor \beta, && (3.2.17) \\ Q_\Sigma(\mathfrak{L}', \Xi, \sigma) &= Q_\Sigma(\mathfrak{L}, \Xi, \sigma) + \oint_{\partial\Sigma} (\xi \rfloor \beta). && (3.2.18) \end{aligned}$$

Of course, these results will only have a local validity in general. In the sequel, though, we shall either deal with equivalent Lagrangians differing from each other by *global* boundary terms, or else our considerations will not rely on globality issues.

Remark 3.2.14. As $E(\mathfrak{L}, \Xi, \sigma)$ can be locally written as

$$E(\mathfrak{L}, \Xi, \sigma) = E^\mu(\mathfrak{L}, \Xi, \sigma)\, \mathrm{d}s_\mu, \tag{3.2.19}$$

where the E^μ's are the components of a vector density, equation (3.2.11) becomes

$$\partial_\mu E^\mu = 0,$$

which, whenever M is the ordinary space-time manifold and $x^0 =: ct$, reads

$$\frac{\partial \varrho}{\partial t} + \operatorname{div} \vec{\jmath} = 0,$$

where we set $\varrho := E^0/c$ and $\vec{\jmath} := (E^1, E^2, E^3)$. So, whenever we can identify the parameter t with time, we get the usual (differential) *equation of continuity* with density ϱ and current $\vec{\jmath}$.

Remark 3.2.15. Consider a *first order* gauge-natural field theory described by a Lagrangian $\mathfrak{L} \colon J^1P_\lambda \to \bigwedge^m T^*M$. Then, the Noether $E(\mathfrak{L}, \Xi)$ associated with $\mathfrak{L}$ and Ξ is locally given by (3.2.10), where $f_\mathfrak{a}{}^\mu \equiv \partial_\mathfrak{a}{}^\mu L$ [*cf.* (3.1.14)]. Now, since the gauge-natural lift Ξ_λ of Ξ onto P_λ is a projectable vector field, it is always possible to choose fibred coordinates $(x^\mu, y^\mathfrak{a}) = (t, x^i, y^\mathfrak{a})_{i=1}^{m-1}$ on P_λ *adapted* to Ξ_λ, i.e. such that $\Xi_\lambda = \partial/\partial t$. Then, the formal generalized Lie derivative $\tilde{\mathcal{L}}_\Xi y^\mathfrak{a}$ reduces to the formal derivative $y^\mathfrak{a}{}_0$ with respect to the coordinate $x^0 = t$. Consider now an m-dimensional region $D \subseteq M$ contained in the domain of the chart (t, x^i) and assume that D is sliced by a family of $(m-1)$-dimensional hypersurfaces Σ_t at constant t. Then, from (3.2.12) it follows that the conserved quantity associated with Ξ and Σ_t along a solution σ of the field equations is

$$\begin{aligned} Q_{\Sigma_t}(\mathfrak{L}, \Xi, \sigma) &= \textstyle\int_{\Sigma_t} (f_\mathfrak{a}{}^0 \tilde{\mathcal{L}}_\Xi y^\mathfrak{a} - L\xi^0) \circ j^1\sigma \, \mathrm{d}s_0 \\ &= \textstyle\int_{\Sigma_t} (f_\mathfrak{a}{}^0 y^\mathfrak{a}{}_0 - L) \circ j^1\sigma \, \mathrm{d}s_0. \end{aligned}$$

If we identify the coordinate t with time, this formula reproduces the standard ADM energy for first order theories (Arnowitt *et al.* 1959, 1960, 1962), and $H_\mathfrak{L} := f_\mathfrak{a}{}^0 y^\mathfrak{a}{}_0 - L$ is recognized to be the (Lagrangian counterpart of the) *Hamiltonian density* of the theory (see also §5.3).

3.2.1 Energy-momentum tensors

Let P_λ be a gauge-natural bundle of order (r, s) associated with a principal bundle $P(M, G)$ and let $\xi^\mu \partial_\mu + \Xi^\mathcal{A} \rho_\mathcal{A}$ be the local representation of a G-invariant vector field Ξ on P. Then, from (2.1.8), (1.10.6), and hence ultimately from (1.10.3), we see that the gauge-natural Lie derivative of a section $\sigma\colon M \to P_\lambda$ with respect to Ξ is a linear partial differential operator of order r and s in ξ^μ and $\Xi^\mathcal{A}$, respectively. Hence, from (3.2.8) it follows that the Noether current associated with a k-th order gauge-natural field theory on P_λ can be expressed as a linear combination of partial derivatives of ξ^μ and $\Xi^\mathcal{A}$ up to order $r + k - 1$ and $s + k - 1$, respectively.

Suppose now the hypotheses of Theorem 3.2.12 are satisfied. Then, we can choose the systems

$$\xi^\lambda, \nabla_{\mu_1}\xi^\lambda, \ldots, \nabla_{(\mu_1} \cdots \nabla_{\mu_{r+k-1})}\xi^\lambda \quad \text{and} \quad \Xi^\mathcal{A}, \nabla_{\mu_1}\Xi^\mathcal{A}, \ldots, \nabla_{(\mu_1} \cdots \nabla_{\mu_{s+k-1})}\Xi^\mathcal{A}$$

as the generators of the modules spanned by

$$\xi^\lambda, \partial_\mu\xi^\lambda, \ldots, \partial_{\mu_1} \cdots \partial_{\mu_{r+k-1}}\xi^\lambda \quad \text{and} \quad \Xi^\mathcal{A}, \partial_{\mu_1}\Xi^\mathcal{A}, \ldots, \partial_{\mu_1} \cdots \partial_{\mu_{s+k-1}}\Xi^\mathcal{A},$$

respectively. Clearly, the former systems generate the same modules as the latter. As far as linear independence is concerned, it enough to notice that there always exists a coordinate system for which $\nabla_{(\mu_1} \cdots \nabla_{\mu_q)} = \partial_{\mu_1} \cdots \partial_{\mu_q}$ at a given point. Also note that symmetrization is required because of the commutation relation [*cf.* (2.6.24) and (1.5.17)]

$$\nabla_{[\mu}\nabla_{\nu]}\xi^\rho = \frac{1}{2}(R^\rho{}_{\sigma\mu\nu}\xi^\sigma - \tau^\sigma{}_{\mu\nu}\nabla_\sigma\xi^\rho) \tag{3.2.20}$$

and the analogous ones for the $\Xi^\mathcal{A}$'s and/or higher order covariant derivatives.

Let us now concentrate on the case $r = 1 = k$, $s = 0$, which encompasses many natural field theories, e.g. any first order tensor field theory (*cf.* Example 1.10.15). For simplicity's sake, in the remainder of this section we shall also assume that the given linear connection (morphism) on M is torsionless. Thus, from the above considerations we know that, if $E(\mathfrak{L}, \Xi) = E(\mathfrak{L}, \xi)$ denotes the Noether current of the theory, then the following decomposition must hold

$$E(\mathfrak{L}, \xi) = (E^{\lambda}{}_{\mu}\xi^{\mu} + E^{\lambda\sigma}{}_{\mu}\nabla_{\sigma}\xi^{\mu})\,\mathrm{d}s_{\lambda}, \tag{3.2.21}$$

where, by construction, $(E^{\lambda}{}_{\mu})$ and $(E^{\lambda\sigma}{}_{\mu})$ are the components of two tensor densities of weight $+1$ known as the ***canonical energy-momentum tensor densities***. For the same reasons, from (3.2.7) it follows that the work form must be expressible as

$$W(\mathfrak{L}, \xi) = (W_{\mu}\xi^{\mu} + W^{\sigma}{}_{\mu}\nabla_{\sigma}\xi^{\mu} + W^{\lambda\sigma}{}_{\mu}\nabla_{(\lambda}\nabla_{\sigma)}\xi^{\mu})\,\mathrm{d}s, \tag{3.2.22}$$

where again, by construction, (W_{μ}), $(W^{\lambda}{}_{\mu})$ and $(W^{\lambda\sigma}{}_{\mu} \equiv W^{(\lambda\sigma)}{}_{\mu})$ are the components of three tensor densities of weight $+1$ known as the ***stress energy-momentum tensor densities***. Decompositions (3.2.21) and (3.2.22) [for arbitrary k, r, s] underlie the proof of Theorem 3.2.12. But, now, from (3.2.21), (1.8.14) and (1.1.2a) it follows that

$$\begin{aligned} \mathrm{d_H}E(\mathfrak{L}, \xi) &= \mathrm{d}_{\lambda}(E^{\lambda}{}_{\mu}\xi^{\mu} + E^{\lambda\sigma}{}_{\mu}\nabla_{\sigma}\xi^{\mu})\,\mathrm{d}s \\ &= [(\nabla_{\lambda}E^{\lambda}{}_{\mu} + \tfrac{1}{2}E^{\lambda\sigma}{}_{\nu}R^{\nu}{}_{\mu\lambda\sigma})\xi^{\mu} + (E^{\sigma}{}_{\mu} + \nabla_{\lambda}E^{\lambda\sigma}{}_{\mu})\nabla_{\sigma}\xi^{\mu} + E^{\lambda\sigma}{}_{\mu}\nabla_{(\lambda}\nabla_{\sigma)}\xi^{\mu}]\,\mathrm{d}s \end{aligned} \tag{3.2.23}$$

where we used (3.2.20) with $\tau = 0$ and replaced formal derivatives with (formal) covariant derivatives since

$$\partial_{\mu}V^{\mu} = \partial_{\mu}V^{\mu} + \Gamma^{\mu}{}_{\nu\mu}V^{\nu} - \Gamma^{\nu}{}_{\nu\mu}V^{\mu} \equiv \nabla_{\mu}V^{\mu} \tag{3.2.24}$$

for any vector density V of weight $+1$ and any symmetric connection Γ. On using (3.2.7), (3.2.23) and (3.2.22), and taking into account the arbitrariness of the ξ^{μ}'s, we then find

$$W_{\mu} = \nabla_{\lambda}E^{\lambda}{}_{\mu} + \tfrac{1}{2}E^{\lambda\sigma}{}_{\nu}R^{\nu}{}_{\mu\lambda\sigma}, \tag{3.2.25a}$$

$$W^{\sigma}{}_{\mu} = E^{\sigma}{}_{\mu} + \nabla_{\lambda}E^{\lambda\sigma}{}_{\mu}, \tag{3.2.25b}$$

$$W^{\lambda\sigma}{}_{\mu} = E^{(\lambda\sigma)}{}_{\mu}. \tag{3.2.25c}$$

Suppose now that our configuration bundle is the fibred product (over M) of a natural bundle $\mathscr{F}M$, coordinatized by $(x^{\lambda}, y^{\boldsymbol{\mu}})$, and the symmetric tensor product $\bigvee^2 T^*M$, coordinatized by $(x^{\lambda}, g_{\mu\nu})$: in other words, a (non-degenerate) section of $\bigvee^2 T^*M$ is a metric on M. Then, identity (3.2.7) takes the form

$$\mathrm{d}_{\lambda}(-L\xi^{\lambda} + f_{\boldsymbol{\mu}}{}^{\lambda}\pounds_{\xi}y^{\boldsymbol{\mu}} + f^{\mu\nu\lambda}\pounds_{\xi}g_{\mu\nu})\,\mathrm{d}s = -e_{\boldsymbol{\mu}}(\mathfrak{L})\pounds_{\xi}y^{\boldsymbol{\mu}} - e^{\mu\nu}(\mathfrak{L})\pounds_{\xi}g_{\mu\nu} \tag{3.2.26}$$

where we took into account (3.2.10) and (3.2.9). Of course, if the sections of $\mathscr{F}M$ were the only dynamical variables, then

$$\mathrm{d}_{\lambda}(-L\xi^{\lambda} + f_{\boldsymbol{\mu}}{}^{\lambda}\pounds_{\xi}y^{\boldsymbol{\mu}})\,\mathrm{d}s = -e_{\boldsymbol{\mu}}(\mathfrak{L})\pounds_{\xi}y^{\boldsymbol{\mu}}.$$

For convenience's sake, in the remainder of this section we shall denote by $E(\mathfrak{L}, \xi)$ and $W(\mathfrak{L}, \xi)$ the $\mathscr{F}M$-part *only* of the Noether current and the work form, respectively, i.e.

$$E(\mathfrak{L}, \xi) := (-L\xi^{\lambda} + f_{\boldsymbol{\mu}}{}^{\lambda}\pounds_{\xi}y^{\boldsymbol{\mu}})\,\mathrm{d}s_{\lambda},$$
$$W(\mathfrak{L}, \xi) := \mathrm{d}_{\lambda}(-L\xi^{\lambda} + f_{\boldsymbol{\mu}}{}^{\lambda}\pounds_{\xi}y^{\boldsymbol{\mu}})\,\mathrm{d}s.$$

Similarly, the symbol '$\approx$' will be used for identities holding modulo $e_{\boldsymbol{\mu}}(\mathfrak{L}) \circ j^1\sigma = 0$ only. Of course, identities (3.2.25) will still be valid, but, in general, (3.2.11) will *not*, unless ξ is a Killing vector field for g (*cf.* §B.1). Indeed, (3.2.26) can now be rewritten as

$$\mathrm{d_H}E(\mathfrak{L}, \xi) \approx -e^{\mu\nu}(\mathfrak{L})\pounds_\xi g_{\mu\nu} - \mathrm{d}_\lambda(f^{\mu\nu\lambda}\pounds_\xi g_{\mu\nu})\,\mathrm{d}s, \tag{3.2.27}$$

whence the statement is clear. Now, recall (B.1.4) and expand the r.h.s. of (3.2.27) to get

$$\mathrm{d_H}E(\mathfrak{L}, \xi) \approx \left[\frac{1}{2}H^{\lambda\sigma}{}_\nu R^\nu{}_{\mu\lambda\sigma}\xi^\mu + (H^\sigma{}_\mu + \nabla_\lambda H^{\lambda\sigma}{}_\mu)\nabla_\sigma\xi^\mu + H^{\lambda\sigma}{}_\mu\nabla_{(\lambda}\nabla_{\sigma)}\xi^\mu\right]\mathrm{d}s, \tag{3.2.28}$$

where we set

$$\begin{aligned} H^{\mu\nu} &:= -2e^{\mu\nu}(\mathfrak{L}) \equiv -2\tfrac{\delta L}{\delta g_{\mu\nu}} \equiv 2g^{\mu\alpha}g^{\nu\beta}\tfrac{\delta L}{\delta g^{\alpha\beta}} \equiv H^{(\mu\nu)}, \\ H^{\lambda\mu\nu} &:= -2f^{\mu\nu\lambda} \equiv -2\tfrac{\partial L}{\partial g_{\mu\nu,\lambda}} \equiv H^{\lambda(\mu\nu)}, \end{aligned} \tag{3.2.29}$$

and used (3.2.24) and (3.2.20), '∇' hereafter denoting the (formal) covariant derivative operator associated with the Levi-Civita connection (morphism) relative to g. Thus, combining (3.2.28) with (3.2.25) yields

$$\tfrac{1}{2}H^{\lambda\sigma}{}_\nu R^\nu{}_{\mu\lambda\sigma} \approx W_\mu \quad \equiv \nabla_\lambda E^\lambda{}_\mu + \tfrac{1}{2}E^{\lambda\sigma}{}_\nu R^\nu{}_{\mu\lambda\sigma}, \tag{3.2.30a}$$

$$H^\sigma{}_\mu + \nabla_\lambda H^{\lambda\sigma}{}_\mu \approx W^\sigma{}_\mu \quad \equiv E^\sigma{}_\mu + \nabla_\lambda E^{\lambda\sigma}{}_\mu, \tag{3.2.30b}$$

$$H^{\lambda\sigma}{}_\mu \approx W^{\lambda\sigma}{}_\mu \equiv E^{(\lambda\sigma)}{}_\mu. \tag{3.2.30c}$$

By definition, $(H^{\mu\nu})$ and $(H^{\lambda\mu\nu})$ are the components of two tensor densities of weight $+1$ known as the (***Hilbert***) ***energy-momentum tensor densities***. Of course, since now we have a metric g at our disposal, we could divide all the tensor densities introduced so far by $\sqrt{g}$, i.e. the square root of the absolute value of the determinant of g, thereby transforming each of them into a tensor field[6]. In particular, $(T^{\mu\nu} := H^{\mu\nu}/\sqrt{g})$ are the components of the well-known (***Hilbert***) ***energy-momentum tensor*** (***field***), whilst $(t^\lambda{}_\mu := E^\lambda{}_\mu/\sqrt{g})$ and $(t^{\lambda\sigma}{}_\mu := E^{\lambda\sigma}{}_\mu/\sqrt{g})$ are the components of the ***first*** and the ***second canonical energy-momentum tensor*** (***field***), respectively.

Now, from (3.2.30*b*) we deduce that

$$H^\sigma{}_\mu \approx E^\sigma{}_\mu + \nabla_\lambda E^{\lambda\sigma}{}_\mu - \nabla_\lambda H^{\lambda\sigma}{}_\mu. \tag{3.2.31}$$

Furthermore, from the *first Bianchi identity*,

$$R^\alpha{}_{[\beta\mu\nu]} \equiv 0,$$

it follows that we can rewrite (3.2.30*a*) as

$$\nabla_{[\lambda}\nabla_{\sigma]}H^{\lambda\sigma}{}_\mu \approx -\nabla_\lambda E^\lambda{}_\mu + \nabla_{[\lambda}\nabla_{\sigma]}E^{\lambda\sigma}{}_\mu. \tag{3.2.32}$$

[6]This amounts to choosing a volume form ${}^g\Sigma$ on M. Locally, ${}^g\Sigma \equiv \sqrt{g}\,\mathrm{d}s$.

Then, on covariantly differentiating (3.2.30*b*) and using (3.2.30*c*) and (3.2.32), we immediately find

$$\nabla_\nu H^{\mu\nu} \approx 0 \approx \nabla_\nu T^{\mu\nu}, \tag{3.2.33}$$

i.e. the first Hilbert energy-momentum tensor (density) is conserved on shell.

Now, we would like to express $H^{\mu\nu}$ in terms of the canonical energy-momentum densities only. To this end, we shall repeat the procedure which led to (3.2.30) on using the triple $(x^\lambda, g^{\mu\nu}, \Gamma^\alpha{}_{\beta\rho})$ instead of $(x^\lambda, g_{\mu\nu}, g_{\alpha\beta,\rho})$ as coordinates for $J^1 \bigvee^2 T^*M$, where the $\Gamma^\alpha{}_{\beta\rho}$'s denote the Christoffel symbols associated with g. This is, of course, always possible since, as is well known, the transformation rule between the first derivatives of g and the Christoffel symbols is invertible. Thus, on starting from

$$\mathrm{d_H} E(\mathcal{L}, \xi) \approx -\left(\frac{\partial L}{\partial g^{\mu\nu}} \pounds_\xi g^{\mu\nu} + \frac{\partial L}{\partial \Gamma^\lambda{}_{\mu\nu}} \pounds_\xi \Gamma^\lambda{}_{\mu\nu}\right) \mathrm{d}s, \tag{3.2.34}$$

and using (B.1.4) and (2.6.29), one easily finds

$$\tfrac{\partial L}{\partial \Gamma^\lambda{}_{\rho\sigma}} R^\lambda{}_{\rho\sigma\mu} \approx \nabla_\lambda E^\lambda{}_\mu + \tfrac{1}{2} E^{\lambda\sigma}{}_\nu R^\nu{}_{\mu\lambda\sigma}, \tag{3.2.35a}$$

$$2\tfrac{\partial L}{\partial g^{\sigma\mu}} \approx E_{\sigma\mu} + \nabla_\lambda E^\lambda{}_{\sigma\mu}, \tag{3.2.35b}$$

$$-\tfrac{\partial L}{\partial \Gamma^\mu{}_{\lambda\sigma}} \approx E^{(\lambda\sigma)}{}_\mu, \tag{3.2.35c}$$

indices having been lowered and raised (here and in the sequel) by means of $(g_{\mu\nu})$ and its inverse $(g^{\nu\rho})$, respectively. Now, from the well-known relation

$$\Gamma^\gamma{}_{\alpha\beta} = \frac{1}{2} g^{\gamma\delta}(\partial_\alpha g_{\delta\beta} + \partial_\beta g_{\alpha\delta} - \partial_\delta g_{\alpha\beta}) \equiv \Gamma^\gamma{}_{(\alpha\beta)}$$

it follows that

$$\begin{aligned} H^{\lambda\mu\nu} &\equiv -2\tfrac{\partial L}{\partial g_{\mu\nu,\lambda}} = -2\tfrac{\partial L}{\partial \Gamma^\gamma{}_{\alpha\beta}} \tfrac{\partial \Gamma^\gamma{}_{\alpha\beta}}{\partial g_{\mu\nu,\lambda}} \\ &= -\tfrac{\partial L}{\partial \Gamma^\gamma{}_{\alpha\beta}} (g^{\gamma\mu}\delta^\nu{}_\beta\delta^\lambda{}_\alpha + g^{\gamma\nu}\delta^\mu{}_\alpha\delta^\lambda{}_\beta - g^{\gamma\lambda}\delta^\mu{}_\alpha\delta^\nu{}_\beta). \end{aligned} \tag{3.2.36}$$

Combining (3.2.35*c*) with (3.2.36) then gives

$$H^{\lambda\mu\nu} \approx E^{(\lambda\nu)\mu} + E^{(\lambda\mu)\nu} + E^{(\mu\nu)\lambda}.$$

Finally, substituting this expression into (3.2.31) yields the desired relation

$$\begin{aligned} H^{\mu\nu} &\approx E^{\mu\nu} + \nabla_\lambda E^{\lambda\mu\nu} - \nabla_\lambda (E^{(\lambda\nu)\mu} + E^{(\lambda\mu)\nu} + E^{(\mu\nu)\lambda}) \\ &\equiv E^{\mu\nu} + \nabla_\lambda (E^{\lambda[\mu\nu]} + E^{\mu[\nu\lambda]} - E^{\nu[\lambda\mu]}) \end{aligned}$$

or, equivalently,

$$T^{\mu\nu} \approx t^{\mu\nu} + \nabla_\lambda (t^{\lambda[\mu\nu]} + t^{\mu[\nu\lambda]} - t^{\nu[\lambda\mu]}), \tag{3.2.37}$$

which is nothing but the ***Belinfante-Rosenfeld formula*** (Belinfante 1940; Rosenfeld 1940). The derivation we have presented here is similar to Kijowski & Tulczyjew's (1979). For a generalization to arbitrary r we refer the interested reader to Gotay & Marsden (1992).

Remark 3.2.16. Note that formulae like (3.2.33) and (3.2.37) hold also for generic s since all the identities concerning the Hilbert energy-momentum tensor only involve the *natural* part of the theory. In other words, they only involve the coefficients of the ξ^μ's, not of the $\Xi^{\mathcal{A}}$'s.

Remark 3.2.17. Earlier on we noted that, in general, (3.2.11) does not hold, unless ξ is a Killing vector field for g, owing to (3.2.27). This happens, of course, because the procedure leading to (3.2.11) holds only *whenever* all *fields and* all *Lagrangians are taken into account*. Indeed, the metric "enters" our Lagrangian $\mathfrak{L}$ by assumption, *but* its behaviour (i.e. *dynamics*) is *not* described by $\mathfrak{L}$. In gravitation theory, gravity is usually described by the Einstein-Hilbert Lagrangian, while the matter fields are described by a suitable "matter Lagrangian", e.g. the Lagrangian $\mathfrak{L}$ of the present case. According to the *principle of minimal coupling*, the total Lagrangian of the theory is then the sum of the Einstein-Hilbert and the matter Lagrangian: it is this "total Lagrangian" the one for which (3.2.11) always holds. In fact, in §3.3.5 we shall see that the Noether current associated with the Einstein-Hilbert Lagrangian $\mathfrak{L}_{\mathrm{EH}}$ reads

$$E(\mathfrak{L}_{\mathrm{EH}}, \xi) = -\frac{1}{\kappa}\xi^\nu G^\mu{}_\nu \sqrt{g}\,\mathrm{d}s_\mu + \mathrm{d}_{\mathrm{H}} U(\mathfrak{L}_{\mathrm{EH}}, \xi),$$

($G_{\mu\nu} \equiv G_{(\mu\nu)}$) denoting the components of the (formal) Einstein tensor, $U(\mathfrak{L}_{\mathrm{EH}}, \xi)$ the superpotential associated with $\mathfrak{L}_{\mathrm{EH}}$, and $\kappa := 8\pi G/c^4$ (*cf.* §3.3.5). Hence, combining this expression with (3.2.26) gives

$$\begin{aligned}\mathrm{d}_{\mathrm{H}} E(\mathfrak{L} + \mathfrak{L}_{\mathrm{EH}}, \xi) &\equiv \mathrm{d}_\lambda(-L\xi^\lambda + f_{\boldsymbol{\mu}}{}^\lambda \pounds_\xi y^{\boldsymbol{\mu}} + f^{\mu\nu\lambda}\pounds_\xi g_{\mu\nu})\,\mathrm{d}s + \mathrm{d}_{\mathrm{H}} E(\mathfrak{L}_{\mathrm{EH}}, \xi) \\ &= -e_{\boldsymbol{\mu}}(\mathfrak{L})\pounds_\xi y^{\boldsymbol{\mu}} - e^{\mu\nu}(\mathfrak{L})\pounds_\xi g_{\mu\nu} - \tfrac{1}{\kappa}\nabla_\mu(\xi^\nu G^\mu{}_\nu)\sqrt{g}\,\mathrm{d}s \\ &= -e_{\boldsymbol{\mu}}(\mathfrak{L})\pounds_\xi y^{\boldsymbol{\mu}} + \left(T^{\mu\nu} - \tfrac{1}{\kappa}G^{\mu\nu}\right)\nabla_\mu\xi_\nu\sqrt{g}\,\mathrm{d}s,\end{aligned}$$

where we used (1.8.15), (3.2.24), (3.2.29) and the *contracted Bianchi identity*

$$\nabla_\nu G^{\mu\nu} \equiv 0.$$

Hence, by virtue of Einstein's equations,

$$G^\mu{}_\nu \approx \kappa\, T^\mu{}_\nu,$$

and the "matter" field equations $e_{\boldsymbol{\mu}}(\mathfrak{L}) \approx 0$,

$$\mathrm{d}_{\mathrm{H}} E(\mathfrak{L} + \mathfrak{L}_{\mathrm{EH}}, \xi) \approx 0,$$

as claimed. □

To conclude this section,we briefly indicate what happens when one includes a tetrad (or variations thereof: *cf.* §2.6 and Chapter 4) as opposed to a metric among the field variables. In this case, one simply defines the (Hilbert) energy-momentum tensor density as

$${}^\theta H^\mu{}_a := -e_a{}^\mu(\mathfrak{L}) \equiv -\frac{\delta L}{\delta\theta^a{}_\mu},$$

and the above procedure goes through unmodified. Of course, ${}^{\theta}H^{\mu\nu} := g^{\nu\rho}\theta^a{}_\rho{}^{\theta}H^\mu{}_a$ will not be symmetric in general. Yet, when a tetrad appears in the Lagrangian *only* through the metric, then it is known that ${}^{\theta}H^{\mu\nu}$ is symmetric. Indeed, on using (2.6.8) we get

$$ {}^{\theta}H^{\mu\nu} = -g^{\nu\rho}\theta^a{}_\rho \frac{\delta L}{\delta\theta^a{}_\mu} = -g^{\nu\rho}\theta^a{}_\rho \frac{\delta L}{\delta g_{\alpha\beta}}\frac{\partial g_{\alpha\beta}}{\partial\theta^a{}_\mu} = \frac{1}{2}g^{\nu\rho}\theta^a{}_\rho H^{\alpha\beta}(\eta_{ab}\theta^b{}_\beta\delta^\mu{}_\alpha + \eta_{ba}\theta^b{}_\alpha\delta^\mu{}_\beta) = H^{\mu\nu}, \tag{3.2.38} $$

which shows that ${}^{\theta}H^{\mu\nu}$ is symmetric (and its definition consistent). A *caveat*, though: despite the fact that (3.2.38) is formally correct, it is dangerous in general to try to gain information about the conserved quantities, associated with a Lagrangian whose primary variable is a tetrad, from the metric, *even if* the former enters the Lagrangian only through the latter. This is because the definition of a Noether current involves explicitly a Lie derivative [formula (3.2.8)], which is, crucially, a category-dependent operator (Remark 2.1.5), and, whereas the metric is (usually regarded as) a natural object, a tetrad is not necessarily—in particular, in the Einstein (-Cartan) -Dirac theory the (spin-) tetrad in question *cannot* be considered as such (*cf.* Chapter 4).

3.3 Examples

We shall now give a number of examples to illustrate how Noether's theorem is applied to some important field theories in actual fact. Except for classical mechanics, we shall always assume that the given base manifold M can be equipped with a pseudo-Riemannian metric g (*cf.* §B.1). In particular, we shall assume that Proca, Yang-Mills and Maxwell fields are *dynamically* coupled with g, whose dynamics is then described by the standard Einstein-Hilbert Lagrangian, as per §3.3.5. Therefore, we should more properly speak of "Einstein-Proca", "Einstein-Yang-Mills" and "Einstein-Maxwell" field theories.

3.3.1 Classical mechanics

In classical mechanics the configuration bundle is (diffeomorphic to) the trivial[7] bundle $(\mathbb{R}\times Q, \mathbb{R}, \mathrm{pr}_1; Q)$, where Q is a manifold. Since any trivial bundle can be regarded as a suitable $\{e\}$-bundle (*cf.*, e.g., Steenrod 1951, §4.1), and hence as a bundle associated with a suitable principal $\{e\}$-bundle (*cf.*, e.g., Kolář *et al.* 1993, §10.8), classical mechanics is indeed—in this minimal sense, at least—a gauge-natural field theory. Somewhat more interestingly, if there is a group G acting on Q, as is often the case, then $\mathbb{R}\times Q$ can be regarded as a G-bundle, and the Lagrangian is required to be invariant with respect to the (lifted) G-action—again, a gauge-natural setting. In any case, the Lagrangian is assumed to be first order: explicitly,

$$ \begin{cases} \mathfrak{L}\colon J^1(\mathbb{R}\times Q) \cong \mathbb{R}\times TQ \to T^*\mathbb{R} \\ \mathfrak{L}\colon \big(t,(q^i,u^i)\big) \mapsto \mathfrak{L}\left(t,(q^i,u^i)\right) =: L(t,q^i,u^i)\,\mathrm{d}t \end{cases}. $$

[7] A well-known topology theorem states that every bundle over a contractible base is trivial (*cf.*, e.g., Steenrod 1951, Corollary 11.6).

Given a projectable vector field $\Xi = \partial/\partial t + \Xi^i \partial/\partial q^i$ on $\mathbb{R}\times Q$, symmetry condition (3.2.3) reads

$$\begin{aligned} \mathrm{d}_t L &= \frac{\partial L}{\partial q^i}\tilde{\mathcal{L}}_\Xi q^i + \frac{\partial L}{\partial u^i}\tilde{\mathcal{L}}_\Xi u^i \\ &\equiv \frac{\partial L}{\partial q^i}\tilde{\mathcal{L}}_\Xi q^i + \frac{\partial L}{\partial u^i}\mathrm{d}_t\tilde{\mathcal{L}}_\Xi q^i, \end{aligned} \tag{3.3.1}$$

where $\mathrm{d}_t \equiv \mathrm{d}/\mathrm{d}t$ denotes the formal derivative on $J^1(\mathbb{R}\times Q) \cong \mathbb{R}\times TQ$ and we used (2.1.14). On specializing (2.1.11′) to the present context, it is immediate to see that $\tilde{\mathcal{L}}_\Xi q^i = u^i - \Xi^i$, and hence (3.3.1) is equivalent to

$$\frac{\partial L}{\partial t} + \Xi^i \frac{\partial L}{\partial q^i} + \mathrm{d}_t \Xi^i \frac{\partial L}{\partial u^i} = 0. \tag{3.3.2}$$

Now, the first order jet prolongation of Ξ is

$$\tilde{\Xi} := J^1\Xi \equiv \frac{\partial}{\partial t} + \Xi^i \frac{\partial}{\partial q^i} + \mathrm{d}_t \Xi^i \frac{\partial}{\partial u^i}, \tag{3.3.3}$$

consistently with formula (1.9.3). On comparing (3.3.3) to (3.3.2), we see that the projectable vector field Ξ is a Lagrangian symmetry iff $\tilde{\Xi}(L) = 0$. The conserved quantity associated with $\mathfrak{L}$ is given by formula (3.2.10) and turns out to be

$$\begin{aligned} E(\mathfrak{L}, \Xi) &= \tfrac{\partial L}{\partial u^i}\tilde{\mathcal{L}}_\Xi q^i - L \\ &\equiv \tfrac{\partial L}{\partial u^i}(u^i - \Xi^i) - L. \end{aligned} \tag{3.3.4}$$

On evaluating (3.3.4) on a critical section $\sigma \colon t \mapsto (t, \sigma^i(t) := q^i \circ \sigma(t))$, one recovers the well-known conservation law

$$\tilde{\Xi}(L) = 0 \implies \frac{\mathrm{d}}{\mathrm{d}t} E(\mathfrak{L}, \Xi, \sigma) = 0.$$

In particular, $\Xi = \partial/\partial t$ is a Lagrangian symmetry iff $\partial L/\partial t = 0$, and the associated conserved current is

$$E\left(\mathfrak{L}, \frac{\partial}{\partial t}\right) = \frac{\partial L}{\partial u^i} u^i - L,$$

which is nothing but the so-called "generalized energy" of the system. Indeed, if the system is conservative, then $E = T + V$, where T and V are the kinetic and potential energy, respectively.

3.3.2 Scalar fields

Let M be an m-dimensional manifold admitting pseudo-Riemannian metrics on itself. A collection $\{y^{\mathfrak{a}}\}_{\mathfrak{a}=1}^n$ of n ($\mathbb{R}$-valued) scalar fields on M can be regarded as a section of the (natural) trivial bundle $M \times \mathbb{R}^n$ over M. Then, Lie derivative (2.1.11′) is simply

$$\tilde{\mathcal{L}}_\Xi y^{\mathfrak{a}} = \mathcal{L}_\xi y^{\mathfrak{a}} \equiv \xi^\nu y^{\mathfrak{a}}{}_\nu,$$

which, substituted into (3.2.10), gives

$$E(\mathfrak{L}, \Xi) = E(\mathfrak{L}, \xi) \equiv \xi^\nu t^\mu{}_\nu \sqrt{g}\, \mathrm{d}s_\mu, \tag{3.3.5}$$

for any first order (natural) Lagrangian $\mathfrak{L}$ on $M \times \mathbb{R}^n$. Here,

$$t^\mu{}_\nu \equiv \frac{1}{\sqrt{g}} \left(\frac{\partial L}{\partial y^{\mathfrak{a}}{}_\mu} y^{\mathfrak{a}}{}_\nu - L \delta^\mu{}_\nu \right) \tag{3.3.6}$$

is the (first) canonical energy-momentum tensor defined in §3.2.1. Since the second canonical energy-momentum tensor is identically zero [compare (3.3.5) to (3.2.21)], from (3.2.37) it follows that the canonical energy-momentum tensor coincides with the (Hilbert) energy-momentum tensor and, hence, is covariantly conserved because of (3.2.33), and symmetric because of (3.2.29)—an attribute that is hardly obvious from (3.3.6).

3.3.3 Proca fields

As in the previous section, M shall denote an m-dimensional manifold admitting pseudo-Riemannian metrics on itself. The first simple example of a theory admitting a non-everywhere vanishing superpotential is the one describing a *constrained* spin 1 massive Boson field α on M, the so-called "Proca field". If $(x^\lambda, y_\mu, y_{\mu\nu})$ denote local natural coordinates on J^1T^*M, the ***Proca Lagrangian*** is the fibre-preserving morphism

$$\begin{cases} \mathfrak{L}_\mathrm{P} \colon \bigvee^2 T^*M \times_M J^1T^*M \to \bigwedge^4 T^*M \\ \mathfrak{L}_\mathrm{P} \colon (g, j^1\alpha) \mapsto \mathfrak{L}(g, j^1\alpha) := \left(\alpha_{[\lambda\mu]} \alpha^{[\lambda\mu]} - \frac{1}{2} m^2 \alpha_\lambda \alpha^\lambda \right) \sqrt{g}\, \mathrm{d}s \end{cases}, \tag{3.3.7}$$

where $\alpha_\lambda := y_\lambda \circ j^1\alpha$, $\alpha_{\lambda\mu} := y_{\lambda\mu} \circ j^1\alpha$, m is a real constant, and, as usual, indices are lowered and raised by means of $g_{\mu\nu}$ and its inverse $g^{\nu\rho}$, respectively. The *Proca equations* of motion are obtained by varying $\mathfrak{L}$ with respect to α. They are

$$\partial_\mu (2\sqrt{g}\, \partial^{[\mu} \alpha^{\lambda]}) = -\sqrt{g}\, m^2 \alpha^\lambda. \tag{3.3.8}$$

Remark 3.3.1. Note that, if we take, as we shall, ∇ to be the covariant derivative operator associated with the Levi-Civita connection relative to g, then partial derivatives can be replaced by covariant derivatives both in (3.3.7) and (3.3.8)—thereby eliminating the annoying factor of $\sqrt{g}$ from both sides of the equation, namely

$$\nabla_\mu \nabla^{[\mu} \alpha^{\lambda]} = -\frac{m^2}{2} \alpha^\lambda.$$

Now, the formal generalized Lie derivative of y_λ is simply

$$\begin{aligned} \tilde{\mathcal{L}}_\Xi y_\lambda &\equiv \mathcal{L}_\xi y_\lambda \\ &= \xi^\nu y_{\lambda\nu} + y_\nu \partial_\lambda \xi^\nu \\ &= \xi^\nu y_{\lambda\nu} + y_\nu \mathrm{d}_\lambda \xi^\nu. \end{aligned}$$

Therefore, on applying formula (3.2.10), we have

$$\begin{aligned} E^\mu(\mathcal{L}_\mathrm{P},\Xi) \equiv E^\mu(\mathcal{L}_\mathrm{P},\xi) &= -L_\mathrm{P}\xi^\mu + f^{\lambda\mu}\tilde{\mathcal{L}}_\Xi y_\lambda \\ &= -L_\mathrm{P}\xi^\mu + f^{\lambda\mu}(\xi^\nu y_{\lambda\nu} + y_\nu \mathrm{d}_\lambda \xi^\nu) \\ &= \xi^\nu(f^{\lambda\mu}y_{\lambda\nu} - L_\mathrm{P}\delta^\mu{}_\nu) + f^{\lambda\mu}y_\nu\mathrm{d}_\lambda\xi^\nu \\ &= \xi^\nu[f^{\lambda\mu}y_{\lambda\nu} - \mathrm{d}_\lambda(f^{\lambda\mu}y_\nu) - L_\mathrm{P}\delta^\mu{}_\nu] + \mathrm{d}_\lambda(f^{\lambda\mu}y_\nu\xi^\nu), \end{aligned} \tag{3.3.9}$$

where, of course,

$$f^{\lambda\mu} \equiv \frac{\partial L_\mathrm{P}}{\partial y_{\lambda\mu}} = 2\sqrt{g}\, y^{[\lambda\mu]}. \tag{3.3.10}$$

Hence,

$$\begin{aligned} U(\mathcal{L}_\mathrm{P},\Xi) \equiv U(\mathcal{L}_\mathrm{P},\xi) &:= \tfrac{1}{2} f^{\lambda\mu}y_\nu\xi^\nu\,\mathrm{d}s_{\mu\lambda} \\ &= \tfrac{1}{2}\xi^\nu y_\nu y^{\lambda\mu}\sqrt{g}\,\mathrm{d}s_{\mu\lambda} \end{aligned}$$

is recognized to be the superpotential associated to the Lagrangian $\mathcal{L}$, *provided* the first term on r.h.s. of (3.3.9) vanishes on shell (*cf.* Remark 3.2.17). Let us check that this is actually the case, as we know it must from the general theory. We have

$$\begin{aligned} E^\mu(\mathcal{L}_\mathrm{P},\xi) &= \xi^\nu[f^{\lambda\mu}y_{\lambda\nu} - \mathrm{d}_\lambda(f^{\lambda\mu}y_\nu) - L_\mathrm{P}\delta^\mu{}_\nu] + \mathrm{d}_\lambda U^{\mu\lambda} \\ &= \xi^\nu(f^{\lambda\mu}y_{\lambda\nu} + y_\nu\mathrm{d}_\lambda f^{\mu\lambda} - f^{\lambda\mu}y_{\nu\lambda} - L_\mathrm{P}\delta^\mu{}_\nu) + \mathrm{d}_\lambda U^{\mu\lambda} \\ &= \xi^\nu[\sqrt{g}\,(4y^{[\lambda\mu]}y_{[\lambda\nu]} - m^2y^\mu y_\nu) - L_\mathrm{P}\delta^\mu{}_\nu] + \xi^\nu(\mathrm{d}_\lambda f^{\mu\lambda} + \sqrt{g}\,m^2y^\mu)y_\nu + \mathrm{d}_\lambda U^{\mu\lambda}, \end{aligned}$$

where we used (3.3.10) and set $U^{\mu\lambda} := f^{\lambda\mu}y_\nu\xi^\nu\sqrt{g}$. Now, the term in square brackets is clearly symmetric in μ and ν, and, on taking also into account formula (3.3.20) below, it is easy to see that it equals the energy-momentum tensor density, whilst the second term vanishes on shell because of (3.3.8). Therefore, $U(\mathcal{L}_\mathrm{P},\xi)$ is indeed the superpotential associated with $\mathcal{L}_\mathrm{P}$ (*cf.* Remark 3.2.17).

Remark 3.3.2. On differentiating (3.3.10) and using (3.3.8) and (3.2.24), we can easily verify that

$$\nabla_\lambda\alpha^\lambda = 0.$$

On using this result and the symmetry properties of the Riemann tensor (*cf.* §2.6.2), we could check that equation (3.3.8) can be recast into the form

$$(\nabla_\mu\nabla^\mu + m^2)\alpha_\lambda = -R^\mu{}_\lambda\alpha_\mu,$$

$R^\mu{}_\lambda := g^{\mu\beta}R^\alpha{}_{\beta\alpha\lambda}$ (*cf.* Kijowski & Tulczyjew 1979). In flat space (which entails $R^\mu{}_\lambda = 0$) this reduces to the usual Klein-Gordon (vector) equation. This is why we call the Proca field a "constrained" field.

3.3.4 Yang-Mills and Maxwell fields

Let $P(M,G)$ be principal bundle, g a metric tensor on M, and $\mathcal{G}$ an Ad_G-invariant metric on G, i.e. such that $\mathcal{G}(\mathrm{Ad}_a\xi_e,\mathrm{Ad}_a\eta_e) = \mathcal{G}(\xi_e,\eta_e)$ for all $a\in G$, $\xi_e,\eta_e\in T_eG\cong\mathfrak{g}$. Then,

we can define an interior product of any two $(P \times_{\mathrm{Ad}} \mathfrak{g})$-valued p-forms α and β on M, locally given by
$$\langle\!\langle \alpha, \beta \rangle\!\rangle \equiv \alpha^{\mathcal{A}}{}_{\mu_1 \ldots \mu_p} \beta^{\mathcal{B}}{}_{\nu_1 \ldots \nu_p} \, \mathcal{G}_{\mathcal{A}\mathcal{B}} \, g^{\mu_1 \nu_1} \cdots g^{\mu_p \nu_p}.$$
Now, let $P_\ell := W^{1,1}P \times_\ell \mathcal{A}$ be the bundle of G-connections introduced in Example 1.10.16. Then the first order Lagrangian

$$\mathfrak{L}_{\mathrm{YM}} := -\frac{1}{16\pi} \langle\!\langle F, F \rangle\!\rangle \sqrt{g} \, \mathrm{d}s$$

on $\bigvee^2 T^*M \times_M J^1 P_\ell$, F being the curvature 2-form associated with a G-connection A on P (*cf.* §2.6.2), is called the ***Yang-Mills Lagrangian*** (relative to A). Unlike the Proca and the scalar field Lagrangian above, and the Einstein-Hilbert Lagrangian of the next section, which are all (*purely*) *natural* Lagrangians, the Yang-Mills Lagrangian is a truly *gauge*-natural one.

In order to find the Noether current $E(\mathfrak{L}_{\mathrm{YM}}, \Xi)$ associated with $\mathfrak{L}_{\mathrm{YM}}$, let us first compute its first variation $\delta\mathfrak{L}_{\mathrm{YM}}$ [in the sense of (3.1.10) or (3.1.15)]. We have:

$$\begin{aligned} \delta\mathfrak{L}_{\mathrm{YM}} &= (f_{\mathcal{A}}{}^{\mu\nu} \delta F^{\mathcal{A}}{}_{\mu\nu} + f_{\mu\nu} \delta g^{\mu\nu}) \mathrm{d}s \\ &= (2 f_{\mathcal{A}}{}^{\mu\nu} \, \mathrm{d}_\nu \delta A^{\mathcal{A}}{}_\mu + 2 f_{\mathcal{A}}{}^{\mu\nu} c^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}} \, \delta A^{\mathcal{B}}{}_\mu A^{\mathcal{C}}{}_\nu + f_{\mu\nu} \delta g^{\mu\nu}) \, \mathrm{d}s \\ &= [\mathrm{d}_\nu (2 f_{\mathcal{A}}{}^{\mu\nu} \, \delta A^{\mathcal{A}}{}_\mu) + 2 (f_{\mathcal{A}}{}^{\mu\nu} c^{\mathcal{A}}{}_{\mathcal{B}\mathcal{C}} A^{\mathcal{C}}{}_\nu - \mathrm{d}_\nu f_{\mathcal{B}}{}^{\mu\nu}) \delta A^{\mathcal{B}}{}_\mu + f_{\mu\nu} \delta g^{\mu\nu}] \, \mathrm{d}s, \end{aligned} \tag{3.3.11}$$

where

$$\begin{aligned} f_{\mathcal{A}}{}^{\mu\nu} &:= \tfrac{\partial L_{\mathrm{YM}}}{\partial F^{\mathcal{A}}{}_{\mu\nu}} = -\tfrac{1}{8\pi} F_{\mathcal{A}}{}^{\mu\nu} \sqrt{g} = \tfrac{\partial L_{\mathrm{YM}}}{\partial A^{\mathcal{A}}{}_{\nu,\mu}}, \\ f_{\mu\nu} &:= \tfrac{\partial L_{\mathrm{YM}}}{\partial g^{\mu\nu}} = \tfrac{1}{8\pi} \left(\tfrac{1}{4} F^{\mathcal{A}}{}_{\alpha\beta} F_{\mathcal{A}}{}^{\alpha\beta} g_{\mu\nu} - F^{\mathcal{A}}{}_{\mu\rho} F_{\mathcal{A}\nu}{}^{\rho} \right) \sqrt{g} \equiv \tfrac{\sqrt{g}}{2} T_{\mu\nu}. \end{aligned} \tag{3.3.12}$$

Comparing, now, (3.3.11) with (3.1.12), we realize that[8]

$$e(\mathfrak{L}_{\mathrm{YM}}) = \left(\tfrac{1}{2} T_{\mu\nu} \sqrt{g} \, \bar{\mathrm{d}} g^{\mu\nu} + \tfrac{1}{4\pi} \nabla_\nu F_{\mathcal{A}}{}^{\mu\nu} \sqrt{g} \, \bar{\mathrm{d}} A^{\mathcal{A}}{}_\mu \right) \otimes \mathrm{d}s, \tag{3.3.13a}$$
$$f(\mathfrak{L}_{\mathrm{YM}}) = \tfrac{1}{4\pi} F_{\mathcal{A}}{}^{\mu\nu} \sqrt{g} \, \bar{\mathrm{d}} A^{\mathcal{A}}{}_\mu \otimes \mathrm{d}s_\nu, \tag{3.3.13b}$$

where, in the first equation, '∇' denotes the (formal) covariant derivative operator with respect to the connection on $\bigwedge^2 T^*M \otimes_M (P \times_{\mathrm{Ad}} \mathfrak{g})$ canonically induced by the Levi-Civita connection on M and the G-connection A on P (*cf.* §§1.5 and 2.6.2). The first term on the r.h.s. of (3.3.13a) vanishes on shell by virtue of Einstein's equation (*cf.* Remark 3.2.17). So does the second one because of the *Yang-Mills equations* of motion,

$$\nabla_\nu F_{\mathcal{A}}{}^{\mu\nu} \approx 0.$$

Now, comparison between (3.1.15) and (3.2.8) [or even, in the first order case, between (3.1.12) and (3.2.10)] tells us that we can read off the Noether current associated

[8] In (3.3.11) the vertical vector field Υ on $\bigvee^2 T^*M \times_M P_\ell$ is locally represented by $\delta g^{\mu\nu} \partial_{\mu\nu} + \delta A^{\mathcal{A}}{}_\mu \partial_{\mathcal{A}}{}^\mu$.

with $\mathcal{L}$ from its first variation (3.3.11). Explicitly,

$$
\begin{aligned}
E(\mathcal{L}_{\mathrm{YM}}, \Xi) &= \tfrac{1}{4\pi}\left(\tfrac{1}{4}F^{\mathcal{A}}{}_{\alpha\beta}F_{\mathcal{A}}{}^{\alpha\beta}\xi^{\nu} + F_{\mathcal{A}}{}^{\mu\nu}\pounds_{\Xi}A^{\mathcal{A}}{}_{\mu}\right)\sqrt{g}\,\mathrm{d}s_{\nu} \\
&= \left(\xi^{\nu}T^{\mu}{}_{\nu} + \tfrac{1}{4\pi}\check{\Xi}^{\mathcal{A}}\nabla_{\nu}F_{\mathcal{A}}{}^{\mu\nu}\right)\sqrt{g}\,\mathrm{d}s_{\mu} - \tfrac{1}{4\pi}\nabla_{\nu}(\check{\Xi}^{\mathcal{A}}F_{\mathcal{A}}{}^{\mu\nu}\sqrt{g})\,\mathrm{d}s_{\mu},
\end{aligned}
\tag{3.3.14}
$$

where we used (3.3.12) and (2.6.27) to obtain the first equality, and (3.3.13*b*) to obtain the second one. Now, the first term on the r.h.s. of (3.3.14) vanishes on shell by virtue of the Einstein and the Yang-Mills equations. As for the second term, first recall that for any bivector density U of weight $+1$ and any symmetric connection Γ we have

$$
\nabla_{\nu}U^{\mu\nu} \equiv \partial_{\nu}U^{\mu\nu} + \Gamma^{\mu}{}_{\rho\nu}U^{\rho\nu} + \Gamma^{\nu}{}_{\rho\nu}U^{\mu\rho} - \Gamma^{\rho}{}_{\rho\nu}U^{\mu\nu} = \partial_{\nu}U^{\mu\nu}. \tag{3.3.15}
$$

Hence, the second term on the r.h.s. of (3.3.14) can be regarded as the (formal) divergence of $U^{\mu\nu} = -1/(4\pi)\,\check{\Xi}^{\mathcal{A}}F_{\mathcal{A}}{}^{\mu\nu}\sqrt{g}$, and

$$
U(\mathcal{L}_{\mathrm{YM}}, \Xi) := -\frac{1}{8\pi}\check{\Xi}^{\mathcal{A}}F_{\mathcal{A}}{}^{\mu\nu}\sqrt{g}\,\mathrm{d}s_{\mu\nu} \tag{3.3.16}
$$

is recognized to be the superpotential associated with the Yang-Mills Lagrangian. At first, the seeming indeterminacy of this superpotential might look puzzling (the $\check{\Xi}^{\mathcal{A}}$'s are *arbitrary* functions), so that some have preferred to impose *by hand* the horizontal lift of ξ (*cf.* §1.5), i.e. set $\Xi = \hat{\xi}$, whence, of course,

$$
E(\mathcal{L}_{\mathrm{YM}}, \Xi) = E(\mathcal{L}_{\mathrm{YM}}, \hat{\xi}) \equiv \xi^{\nu}T^{\mu}{}_{\nu}\sqrt{g}\,\mathrm{d}s_{\mu}
$$

and

$$
U(\mathcal{L}_{\mathrm{YM}}, \Xi) = U(\mathcal{L}_{\mathrm{YM}}, \hat{\xi}) \equiv 0
$$

(*cf.*, e.g., Kijowski & Tulczyjew 1979). However, we can convince ourselves that (3.3.16) is the "right answer" by specializing to the electromagnetic case, of which Yang-Mills theory can be thought of as the non-commutative generalization.

Indeed, electromagnetism is just a Yang-Mills theory with structure group U(1), the (Abelian) group of all unimodular complex numbers, i.e. $\mathrm{U}(1) \equiv \{\, e^{i\theta} : \theta \in \mathbb{R} \,\} \cong S^1$ (*cf.*, e.g., Atiyah 1979). Since its Lie algebra $\mathfrak{u}(1) \cong i\mathbb{R}$ (*cf.* §C.2) is 1-dimensional, we can omit Lie algebra indices everywhere. Furthermore, U(1) being Abelian, its unique structure constant is zero. Then, the Lagrangian for electromagnetism, or the ***Maxwell Lagrangian***, is

$$
\mathcal{L}_{\mathrm{M}} := -\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu}\sqrt{g}\,\mathrm{d}s,
$$

where

$$
F = \mathrm{d}A
$$

is the electromagnetic field with components [*cf.* (2.6.25)]

$$
F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu},
$$

and [*cf.* (1.5.2)]

$$
\chi = \mathrm{d}x^{\mu} \otimes (\partial_{\mu} - A_{\mu}(x)\rho).
$$

is the U(1)-connection serving as the electromagnetic potential. Then, the (source-free) *Maxwell equations* read

$$\nabla_\nu F^{\mu\nu} \approx 0,$$

whereas the Noether current is

$$E(\mathfrak{L}_{\mathrm{M}}, \Xi) = \left(\xi^\nu T^\mu{}_\nu + \frac{1}{4\pi}\check{\Xi}\nabla_\nu F^{\mu\nu}\right)\sqrt{g}\,\mathrm{d}s_\mu - \frac{1}{4\pi}\nabla_\nu(\check{\Xi}F^{\mu\nu}\sqrt{g})\,\mathrm{d}s_\mu.$$

Therefore, the superpotential associated with $\mathfrak{L}_{\mathrm{M}}$ turns out to be

$$U(\mathfrak{L}_{\mathrm{M}}, \Xi) := -\frac{1}{8\pi}\check{\Xi}F^{\mu\nu}\sqrt{g}\,\mathrm{d}s_{\mu\nu}. \tag{3.3.17}$$

Now, we know that, in general, a vertical G-invariant vector Υ on a principal bundle $P(M, G)$ transforms via the adjoint representation of G (*cf.* §1.3 and Example 1.10.19). But U(1) is Abelian, and hence $\check{\Xi}$ is unchanged under *gauge transformations*, i.e. vertical automorphisms of the structure bundle. Therefore, without loss of generality, we can choose $\check{\Xi} = 1$. In any (4-dimensional) Lorentzian manifold where it is possible to single out a globally defined timelike vector field and, hence, in particular, in Minkowski spacetime $(\mathbb{R}^4, \eta)$, we can consider the conserved quantity

$$Q(\mathfrak{L}_{\mathrm{M}}, \Xi, \sigma) = \oint_{\partial\Sigma} U(\mathfrak{L}_{\mathrm{M}}, \Xi, \sigma)$$

associated with a (constant time) spacelike region Σ with boundary $\partial\Sigma$ (and a critical section of $\bigvee^2 T^*M \times_M P_\ell$). Then, with our conventions, $(E^i := -F^{0i}(\sigma))_{i=1}^3$ can be identified with the components of the electric field, and from (3.3.17) we obtain[9]

$$Q(\mathfrak{L}_{\mathrm{M}}, \Xi, \sigma) = \frac{1}{4\pi}\oint_{\partial\Sigma} \vec{E}\cdot\vec{n}\,\mathrm{d}S,$$

i.e. precisely the total charge contained in Σ.

3.3.5 General relativity

Let (M, g) be a 4-Lorentzian manifold. In its standard Lagrangian formulation general relativity is described by the ***Einstein-Hilbert Lagrangian***

$$\begin{cases} \mathfrak{L}_{\mathrm{EH}} \colon J^2\bigvee^2 T^*M \to \bigwedge^4 T^*M \\ \mathfrak{L}_{\mathrm{EH}} \colon j^2 g \mapsto \mathfrak{L}_{\mathrm{EH}}(j^2 g) := -\dfrac{1}{2\kappa}\sqrt{g}g^{\mu\nu}R_{\mu\nu}\,\mathrm{d}s, \end{cases} \tag{3.3.18}$$

where $\kappa \equiv 8\pi G/c^4$ and $R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu}$, $(R^\alpha{}_{\mu\alpha\nu})$ being the components of the curvature 2-form (or Riemann tensor) associated with the Levi-Civita connection relative to g (*cf.* §2.6.2). Of course, as we are thinking of g as a dynamical variable whose values will be ultimately determined by the (Einstein) equations of motion, M should be more precisely thought of as a 4-dimensional manifold satisfying all the topological requirements needed

[9] Here, $\vec{E} \equiv E^i\partial_i$, $\vec{n}$ is the outward normal to $\partial\Sigma$, and $\mathrm{d}S$ its surface element.

to admit Lorentzian metrics on itself, so that (M, g) is a Lorentzian manifold for any solution g of the field equations. For the same reasons, the configuration bundle of the theory is, strictly speaking, the (natural vector) bundle $\mathrm{Lor}(M)$ of (non-degenerate) Lorentzian metrics on M rather than simply $\bigvee^2 T^*M$. All this will be hereafter understood.

The variation of $\mathfrak{L}_{\mathrm{EH}}$ is

$$\begin{aligned}\delta\mathfrak{L}_{\mathrm{EH}} &= -\tfrac{1}{2\kappa}\delta(\sqrt{g}g^{\mu\nu}R_{\mu\nu})\,\mathrm{d}s\\ &= -\tfrac{1}{2\kappa}\sqrt{g}(G_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu})\,\mathrm{d}s\\ &= -\tfrac{1}{2\kappa}G_{\mu\nu}\sqrt{g}\,\delta g^{\mu\nu}\,\mathrm{d}s - \mathrm{d}_{\mathrm{H}}\left(\tfrac{1}{2\kappa}g^{\mu\nu}\sqrt{g}\,\delta u^{\alpha}{}_{\mu\nu}\,\mathrm{d}s_{\alpha}\right),\end{aligned} \tag{3.3.19}$$

where we used the relation

$$\delta\sqrt{g} = -\frac{1}{2}\sqrt{g}g_{\mu\nu}\delta g^{\mu\nu} \tag{3.3.20}$$

and set

$$\begin{aligned}G_{\mu\nu} &:= R_{\mu\nu} - \tfrac{1}{2}g^{\rho\sigma}R_{\rho\sigma}g_{\mu\nu},\\ u^{\alpha}{}_{\mu\nu} &:= \Gamma^{\alpha}{}_{\mu\nu} - \tfrac{1}{2}(\delta^{\alpha}{}_{\mu}\Gamma^{\beta}{}_{\beta\nu} + \delta^{\alpha}{}_{\nu}\Gamma^{\beta}{}_{\beta\mu}).\end{aligned}$$

The *Einstein equations* of motion are then

$$G_{\mu\nu} \approx 0$$

or, in the presence of matter fields (*cf.* §3.2.1),

$$G_{\mu\nu} \approx \kappa T_{\mu\nu}.$$

From (3.3.19) (and taking into account the fact that the configuration bundle is a natural vector bundle) we deduce that the Noether current associated with $\mathfrak{L}_{\mathrm{EH}}$ is

$$E(\mathfrak{L}_{\mathrm{EH}}, \Xi) = E(\mathfrak{L}_{\mathrm{EH}}, \xi) \equiv -\xi \,\lrcorner\, \mathfrak{L}_{\mathrm{EH}} - \frac{1}{2\kappa}g^{\mu\nu}\sqrt{g}\,\pounds_{\xi}u^{\alpha}{}_{\mu\nu}\,\mathrm{d}s_{\alpha}.$$

On making use, now, of (2.6.29), after a little algebra we get to the expression

$$E(\mathfrak{L}_{\mathrm{EH}}, \xi) = -\frac{1}{\kappa}\xi^{\nu}G^{\mu}{}_{\nu}\sqrt{g}\,\mathrm{d}s_{\mu} + \mathrm{d}_{\nu}\left(\frac{1}{\kappa}\nabla^{[\mu}\xi^{\nu]}\sqrt{g}\right)\mathrm{d}s_{\mu}, \tag{3.3.21}$$

whence

$$U(\mathfrak{L}_{\mathrm{EH}}, \xi) = \frac{1}{2\kappa}\nabla^{[\mu}\xi^{\nu]}\sqrt{g}\,\mathrm{d}s_{\mu\nu}$$

is recognized to be the superpotential associated with $\mathfrak{L}_{\mathrm{EH}}$, and Remark 3.2.17 is now fully justified. $U(\mathfrak{L}_{\mathrm{EH}}, \xi)$ was originally derived in a Hamiltonian (multisymplectic) framework by Kijowski (1978) (see also Kijowski & Tulczyjew 1979), and is a *half* of the well-known Komar (1959) potential, thereby suffering from similar drawbacks (*cf.*, e.g., Katz 1985).

Note, though, that the Einstein-Hilbert Lagrangian is *degenerate*. Indeed, according to the general theory, the field equations associated with a second order Lagrangian should be fourth order (Proposition 3.1.2), but Einstein equations are clearly second order, *as if* $\mathfrak{L}_{\mathrm{EH}}$ were first order. Indeed, as Einstein himself (1916) realized, it is possible to decompose the Einstein-Hilbert Lagrangian into a (formal) divergence and an equivalent first order Lagrangian (*cf.* Theorem 3.1.3). Explicitly,

$$\mathfrak{L}_{\mathrm{EH}} = \mathfrak{L}_{\mathrm{E}} - \mathrm{d}_\alpha \left(\frac{1}{2\kappa} g^{\mu\nu} u^\alpha{}_{\mu\nu} \sqrt{g} \right) \mathrm{d}s,$$

where

$$\mathfrak{L}_{\mathrm{E}} := -\frac{1}{2\kappa} g^{\mu\nu} (\Gamma^\alpha{}_{\mu\sigma}\Gamma^\sigma{}_{\alpha\nu} - \Gamma^\alpha{}_{\alpha\sigma}\Gamma^\sigma{}_{\mu\nu}) \sqrt{g}\, \mathrm{d}s.$$

Now, we know that, although equivalent Lagrangians give rise, by their very definition, to the same equations of motion, the associated conserved quantities are, in general, different [*cf.* (3.2.18)], so that we could hope that using $\mathfrak{L}_{\mathrm{E}}$ instead of $\mathfrak{L}_{\mathrm{EH}}$ would improve the situation. Unfortunately, since Γ is not a tensor, $\mathfrak{L}_{\mathrm{E}}$ is *not* covariant, and hence nor are the ensuing conserved quantities.

Following an earlier idea of Rosen (1940), we can then introduce a *background* (symmetric linear) connection $\hat{\Gamma}$ in order to obtain a covariant splitting of $\mathfrak{L}_{\mathrm{EH}}$. Explicitly,

$$\mathfrak{L}_{\mathrm{EH}} = \mathfrak{L}_{\mathrm{FF}} - \mathrm{d}_{\mathrm{H}} \left(\frac{1}{2\kappa} g^{\mu\nu} w^\alpha{}_{\mu\nu} \sqrt{g}\, \mathrm{d}s_\alpha \right),$$

where

$$\mathfrak{L}_{\mathrm{FF}} := -\frac{1}{2\kappa} g^{\mu\nu} (\hat{R}_{\mu\nu} + q^\alpha{}_{\mu\sigma} q^\sigma{}_{\alpha\nu} - q^\alpha{}_{\alpha\sigma} q^\sigma{}_{\mu\nu}) \sqrt{g}\, \mathrm{d}s,$$

and we set

$$\begin{aligned} q^\alpha{}_{\rho\sigma} &:= \Gamma^\alpha{}_{\rho\sigma} - \hat{\Gamma}^\alpha{}_{\rho\sigma}, \\ w^\alpha{}_{\mu\nu} &:= u^\alpha{}_{\mu\nu} - \hat{u}^\alpha{}_{\mu\nu}, \\ \hat{u}^\alpha{}_{\mu\nu} &:= \hat{\Gamma}^\alpha{}_{\mu\nu} - \tfrac{1}{2}(\delta^\alpha{}_\mu \hat{\Gamma}^\beta{}_{\beta\nu} + \delta^\alpha{}_\nu \hat{\Gamma}^\beta{}_{\beta\mu}), \end{aligned}$$

$\hat{R}$ denoting the curvature 2-form associated with $\hat{\Gamma}$. Then, from (3.2.17) it follows that the superpotential associated with $\mathfrak{L}_{\mathrm{FF}}$ is

$$U(\mathfrak{L}_{\mathrm{FF}}, \xi) = \frac{1}{2\kappa} (\nabla^{[\mu} \xi^{\nu]} - \xi^{[\mu} w^{\nu]}{}_{\rho\sigma} g^{\rho\sigma}) \sqrt{g}\, \mathrm{d}s_{\mu\nu}. \tag{3.3.22}$$

This derivation is due to Ferraris & Francaviglia (1990), although superpotential (3.3.22) had already been found by Katz (1985) in an *ad hoc* way. The good news is that this superpotential overcomes the drawbacks of Komar's (1959) potential. In particular, with a suitable (but generally obvious) choice of background connection, it reproduces:

- the Schwarzschild mass parameter m_{S} for the Schwarzschild solution,
- the correct mass $m_{\mathrm{S}} - e^2/2r$ for the Reissner-Nordström solution,
- the correct angular momentum $\ell = m_{\mathrm{S}} a$ for the the Kerr solution,
- the ADM mass at spacelike infinity,
- the Bondi mass at null infinity.

Finally, we should mention that quite recently Noether's theorem was successfully used to provide a general definition of entropy for stationary black holes (Fatibene *et al.* 1999*a*, 1999*b*, 2000, and references therein), to obtain the first law of thermodynamics for rigidly rotating horizons (Allemandi *et al.* 2001), and to give a general definition of energy for the N-body problem in $(1+1)$-dimensional gravity (Mann *et al.* 2000).

Chapter 4

Gauge-natural gravitation theory

> L'universo è infinito perché deve consentire assolutamente tutto quello che è permesso, perché tutto quello che è permesso è obbligatorio.
>
> T. Regge, in: P. Levi & T. Regge, *Dialogo*

This chapter is a reformulation of results that first appeared in Godina *et al.* (2001) and Matteucci (2002).

4.1 Motivation

The Einstein-Dirac theory is the (classical) field theory describing a spin 1/2 massive Fermion field minimally coupled with Einstein's gravitational field in a curved space-time. The Einstein-Cartan-Dirac theory is a modification of the Einstein-Dirac theory, which allows for the presence of torsion as a possible "source" of spin (*cf.* §4.3 below).

In this chapter we shall show that the functorial approach of gauge-natural bundles and the general theory of Lie derivatives developed in Chapters 2 and 3 is *essential* for a correct geometrical formulation of the Einstein (-Cartan) -Dirac theory and, at the same time, yields an unexpected *indeterminacy* in the concept of conserved quantities. In the Einstein-Cartan-Dirac case, such an indeterminacy *can be regarded* as the well-known indeterminacy which occurs in gauge theory (Noether 1918; Giachetta *et al.* 1997; Fatibene 1999; Barnich & Brandt 2002), although there are serious conceptual risks involved in dismissing this "metric-affine" theory of gravitation as a standard "gauge theory" (Trautman 1980; Giachetta *et al.* 1997). This is certainly not the case, though, for the Einstein-Dirac theory proper, which can by no means be viewed as such. We shall show that, in both cases, this indeterminacy actually arises from the very fact that, when coupled with Dirac fields, Einstein's general relativity can no longer be regarded as a purely natural theory, because, in order to incorporate spinors, one *must* enlarge the class of morphisms of the theory.

Indeed, it is well-known that there are no representations of the group $\mathrm{GL}(4,\mathbb{R})$ of the automorphisms of $\mathbb{R}^4$ which behave like spinors under the subgroup of Lorentz transformations. Therefore, if one aims at considering the coupling between general relativity and Fermion fields, one is forced to resort to the so-called "tetrad formalism" (*cf.*, e.g., Weinberg 1972). Yet, there seems to have been a widespread misunderstanding of the full mathematical (and physical) significance of this. Leaving all the technicalities to the later sections, it will suffice here to briefly recall how the concept of a tetrad is usually introduced.

On relying on the "principle of equivalence", which mathematically is tantamount to the simple statement that every manifold is locally flat, at every point $\tilde{x}$ of space-time one can erect a set of coordinates (X^a) that are *locally inertial* at $\tilde{x}$. The components of the metric in any general *non-inertial* coordinate system are then[1]

$$g_{\mu\nu}(x) = \theta^a{}_\mu(x)\theta^b{}_\nu(x)\eta_{ab}, \tag{4.1.1}$$

where $\|\eta_{ab}\| := \operatorname{diag}(1, -1, -1, -1)$ (*cf.* §C.1) and

$$\theta^a{}_\mu(\tilde{x}) := \left.\frac{\partial X^a(x)}{\partial x^\mu}\right|_{x=\tilde{x}}, \tag{4.1.2}$$

Thus, if we change our general non-inertial coordinates from (x^μ) to (x'^μ), $\theta^a{}_\mu$ will change according to the rule

$$\theta^a{}_\mu \mapsto \theta'^a{}_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\theta^a{}_\nu. \tag{4.1.3}$$

Therefore, $(\theta^a{}_\mu)$ must be regarded as the components of *four* 1*-forms* (θ^a), *not* of a single tensor field θ. This set of four 1-forms is what is known as a *tetrad*.

At this stage the Latin index a is just a "label" and, for any a, θ^a is indeed a natural object. But the reason why a tetrad was introduced in the first place is precisely that we then wanted to "switch on" that Latin index in order to incorporate spinors into our formalism. This means that $\theta^a{}_\mu$ will have to additionally change according to the rule

$$\theta^a{}_\mu(x) \mapsto L^a{}_b(x)\theta^b{}_\mu(x), \tag{4.1.4}$$

where $L(x)$ is the (space-time-dependent) Lorentz transformation induced (modulo a sign) by a given spinorial transformation S under the group epimorphism $\Lambda\colon \mathrm{Spin}(1,3)^e \to \mathrm{SO}(1,3)^e$ (*cf.* §D.1).

This is precisely the point that has been too often overlooked. Unlike (4.1.3), transformation law (4.1.4) does *not* descend from definition (4.1.2), but is a requirement we have imposed *a posteriori*. In other words, we have *changed the definition* of $\theta^a{}_\mu$ in such a way that now $(\theta^a{}_\mu)$ must be regarded as the components of a *non-natural* object θ.

There is another important point that has been traditionally overlooked, which is of pre-eminent physical significance. Recall, indeed, that spinor fields can be defined on a manifold M only if M admits a "spin structure". Now, the standard definition of a spin structure involves *fixing a metric* on M (*cf.* §D.2), a framework which is certainly well-suited to a situation in which the gravitational field is considered *unaffected* by spinors, but is otherwise unable to describe the complete interaction and feedback between gravity and spinor fields (van den Heuvel 1994; Sławianowski 1996). To this end, the concept of a *free spin structure* must be introduced.

Ultimately, the solution to both the aforementioned problems lies in suitably defining the bundle of which θ is to be a section. This leads to the concept of a *spin-tetrad*, which

[1] Here and in the rest of this chapter both Latin and Greek indices range from 0 to 3.

turns out to be a gauge-natural object (Fatibene *et al.* 1998; Godina *et al.* 2000, 2001; Matteucci 2002).

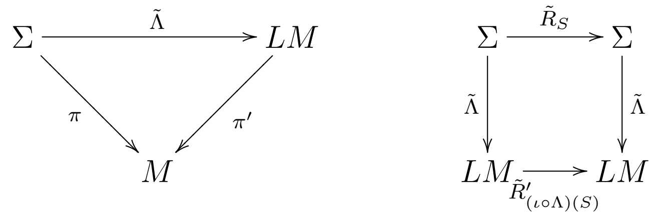
4.2 Spin-tetrads, spin-connections and spinors

To the best of our knowledge, the concept of a "free spin structure" was originally introduced (with a different purpose) by Plymen & Westbury (1987) (see also Swift 1988). It was then rediscovered by van den Heuvel (1994) for the very reason mentioned in §4.1 and further analysed by Fatibene *et al.* (1998) and Fatibene & Francaviglia (1998). The notion of a "spin-tetrad" as a section of a suitable gauge-natural bundle over M was first proposed by Fatibene *et al.* (1998).

Definition 4.2.1. Let M be a 4-dimensional manifold admitting Lorentzian metrics of signature -2, i.e. satisfying the topological requirements which ensure the existence on it of Lorentzian structures [$\mathrm{SO}(1,3)^e$-reductions], and let Λ be the epimorphism which exhibits $\mathrm{Spin}(1,3)^e$ as the twofold covering of $\mathrm{SO}(1,3)^e$. A ***free spin structure*** on M consists of a principal bundle $\pi\colon \Sigma \to M$ with structure group $\mathrm{Spin}(1,3)^e$ and a map $\tilde{\Lambda}\colon \Sigma \to LM$ such that

$$\tilde{\Lambda} \circ \tilde{R}_S = \tilde{R}'_{(\iota\circ\Lambda)(S)} \circ \tilde{\Lambda} \quad \forall S \in \mathrm{Spin}(1,3)^e,$$
$$\pi' \circ \tilde{\Lambda} = \pi,$$

$\tilde{R}$ and $\tilde{R}'$ denoting the canonical right actions on Σ and LM, respectively, $\iota\colon \mathrm{SO}(1,3)^e \to \mathrm{GL}(4,\mathbb{R})$ the canonical injection of Lie groups, and $\pi'\colon LM \to M$ the canonical projection (*cf.* Definition D.2.1). Equivalently, the diagrams



are commutative. We shall call the bundle map $\tilde{\Lambda}$ a *spin-frame* on Σ.

This definition of a spin structure induces metrics on M. Indeed, given a spin-frame $\tilde{\Lambda}\colon \Sigma \to LM$, we can define a metric g via the reduced subbundle $\mathrm{SO}(M,g) := \tilde{\Lambda}(\Sigma)$ of LM. In other words, the (*dynamic*) metric $g \equiv g_{\tilde{\Lambda}}$ is defined to be the metric such that frames in $\tilde{\Lambda}(\Sigma) \subset LM$ are g-orthonormal frames. It is important to stress that in our picture the metric g is built up *a posteriori*, after a spin-frame has been determined by the field equations in a way which is compatible with the (free) spin structure one has used to define spinors.

Now, if we want to regard spin-frames as dynamical variables in a Lagrangian field theory, we should be able to represent them as (global) sections of a suitable configuration bundle. This motivates the following

Definition 4.2.2. Let Λ be as in Definition 4.2.1 and consider the following left action of the group $W_4^{1,0}\mathrm{Spin}(1,3)^e$ on the manifold $\mathrm{GL}(4,\mathbb{R})$

$$\begin{cases} \rho\colon W_4^{1,0}\mathrm{Spin}(1,3)^e \times \mathrm{GL}(4,\mathbb{R}) \to \mathrm{GL}(4,\mathbb{R}) \\ \rho\colon ((\alpha^k{}_l, S^m{}_n), \beta^i{}_j) \mapsto \beta'^i{}_j := (\Lambda(S))^i{}_k \beta^k{}_l \tilde{\alpha}^l{}_j \end{cases}$$

together with the associated bundle $\Sigma_\rho := W^{1,0}\Sigma \times_\rho \mathrm{GL}(4,\mathbb{R})$. Σ_ρ is a fibre bundle associated with $W^{1,0}\Sigma$, i.e. a gauge-natural bundle of order $(1,0)$. A section of Σ_ρ will be called a ***spin-tetrad*** (*cf.* §2.6 for the corresponding metric-dependent definition).

If $(\theta^a{}_\mu)$ denote the components of a spin-tetrad θ in some local chart, then the components $(g_{\mu\nu})$ of the induced metric g in the associated chart read

$$g_{\mu\nu} \equiv \theta^a{}_\mu \theta^b{}_\nu \eta_{ab}, \tag{4.2.1}$$

formally identical with equation (4.1.1), but with *both* (4.1.3) *and* (4.1.4) built in.

Recall now how a G-connection was defined (Example 1.10.16) and consider the following

Definition 4.2.3. Let $\mathfrak{so}(1,3) \cong \mathfrak{spin}(1,3)$ denote the Lie algebra of $\mathrm{SO}(1,3)^e$ and consider the following left action of the group $W_4^{1,1}\mathrm{Spin}(1,3)^e$ on the vector space $(\mathbb{R}^4)^* \otimes \mathfrak{so}(1,3)$

$$\begin{cases} \ell\colon W_4^{1,1}\mathrm{Spin}(1,3)^e \times \Big((\mathbb{R}^4)^* \otimes \mathfrak{so}(1,3)\Big) \to (\mathbb{R}^4)^* \otimes \mathfrak{so}(1,3) \\ \ell\colon ((\alpha^l{}_m, S^n{}_p, S^q{}_{rs}), w^i{}_{jk}) \mapsto w'^i{}_{jk} := [(\Lambda(S))^i{}_l w^l{}_{mn} (\Lambda(S^{-1}))^m{}_j \\ \qquad\qquad - (\Lambda(S))^i{}_{ln} (\Lambda(S^{-1}))^l{}_j] \tilde{\alpha}^n{}_k \end{cases},$$

where we are using the notation of Example 1.10.16 with $a := \Lambda(S(0))$ and $(\Lambda(S))^i{}_{jk} := \Big(\partial_k (a^{-1}\Lambda(S(x))|_{x=0})\Big)^i{}_j$. Clearly, the associated bundle $\Sigma_\ell := W^{1,1}\Sigma \times_\ell \Big((\mathbb{R}^4)^* \otimes \mathfrak{so}(1,3)\Big)$ is a gauge-natural (affine) bundle of order $(1,1)$ isomorphic to the bundle of $\mathrm{SO}(1,3)^e$-connections over M (*cf.* Example 1.10.16). A section of Σ_ℓ will be called a ***spin-connection***.

Note that also spinors can be regarded as sections of a suitable gauge-natural bundle over M. Indeed, if $\hat{\gamma}$ is the linear representation of $\mathrm{Spin}(1,3)^e$ on the vector space $\mathbb{C}^4$ induced by the given choice of γ matrices, then the associated vector bundle $\Sigma_{\hat{\gamma}} := \Sigma \times_{\hat{\gamma}} \mathbb{C}^4$ is a gauge-natural (vector) bundle of order $(0,0)$ whose sections represent *spinors* (or, more precisely, *spin-vector fields*: *cf.* §D.2).

Note also that, in the present picture, the *spinor connection* $\tilde{\omega}$ corresponding to a given spin-connection ω may be defined in terms of ω as

$$\tilde{\omega} := \Big(\mathrm{id} \otimes (\Lambda')^{-1}\Big)(\omega),$$

$\Lambda' := T_e\Lambda$ denoting the Lie algebra isomorphism between $\mathfrak{spin}(1,3)$ and $\mathfrak{so}(1,3)$ [*cf.* (D.2.2)]. Locally, the components $(\tilde{\omega}_\mu)$ of $\tilde{\omega}$ read

$$\tilde{\omega}_\mu \equiv -\frac{1}{4}\omega^{ab}{}_\mu \gamma_{ab},$$

($\omega^a{}_{c\mu} =: \omega^{ab}{}_\mu \eta_{bc}$) denoting the components of ω [*cf.* (D.2.3) and (D.1.2)].

4.3 Riemann-Cartan geometry on spin manifolds

Throughout the rest of this chapter we shall use Cartan's language of vector (bundle)-valued differential forms (on M), which will prove to be an elegant and compact way to express our findings. To this end, let $\Sigma_{\hat\rho} := \Sigma \times_{\hat\rho} \mathbb{R}^4$ denote the vector bundle associated with Σ via the action

$$\begin{cases} \hat\rho\colon \mathrm{Spin}(1,3)^e \times \mathbb{R}^4 \to \mathbb{R}^4 \\ \hat\rho\colon (S^j{}_k, \beta^i) \mapsto \beta'^i := \Lambda(S)^i{}_j \beta^j \end{cases}.$$

Then, a spin-tetrad can be equivalently regarded as a $\Sigma_{\hat\rho}$-valued 1-form on M locally reading

$$\theta := \theta^a \otimes f_a, \quad \theta^a := \theta^a{}_\mu \, \mathrm{d}x^\mu, \tag{4.3.1}$$

(f_a) denoting a local fibre basis of $\Sigma_{\hat\rho}$. Furthermore, by $\mathfrak{gl}(\Sigma_{\hat\rho})$ we shall mean the vector bundle over M given by the value at $\Sigma_{\hat\rho}$ of the canonical extension of the functor $\mathfrak{gl}$ to the category of vector bundles and their homomorphisms (*cf.* Kolář *et al.* 1993, §6.7). Finally, if ($\omega^a{}_{b\mu}$) are the components of a spin-connection in some local chart, it is convenient to introduce the notation (*cf.* §2.6.2)

$$\omega^a{}_b := \omega^a{}_{b\mu} \, \mathrm{d}x^\mu.$$

Now, let '∇' be the covariant derivative operator with respect to the connection on T^p_qM *naturally* induced by a (natural) linear connection Γ on M (*cf.* Example 1.10.17), T^p_qM denoting the (p,q)-tensor bundle over M (*cf.* Example 1.10.15). Classically, Riemann-Cartan geometry is characterized by two conditions: the covariant constancy of the metric,

$$\nabla g = 0, \tag{4.3.2}$$

just as in ordinary Riemannian geometry, and the presence of a (not necessarily zero) torsion tensor τ (*cf.* §1.5.1).

In the present gauge-natural setting we can introduce analogous concepts serving a similar purpose. In particular, if θ is a spin-tetrad in the sense of Definition 4.2.2 and g is the metric induced by θ via (4.2.1), equation (4.3.2) can be derived by the condition

$$\nabla \theta = 0, \tag{4.3.3}$$

where, here, '∇' denotes the covariant derivative operator with respect to the connection on Σ_ρ *canonically* induced by the connections Γ and $\tilde\omega$ on LM and Σ, respectively (*cf.* §2.6). Accordingly, we can define a *torsion 2-form* as the $\Sigma_{\hat\rho}$-valued 2-form, which we shall denote again by τ, given by the expression

$$\tau := \mathcal{D}\theta$$

or, equivalently,

$$\tau := \tau^a \otimes f_a, \quad \tau^a := \mathcal{D}\theta^a \equiv \mathrm{d}\theta^a + \omega^a{}_b \wedge \theta^b, \tag{4.3.4}$$

'$\mathcal{D}$' denoting the covariant exterior derivative operator (*cf.* §1.5) and θ being as in (4.3.1).

Moreover, we can define a *contortion 1-form* as the $\mathfrak{gl}(\Sigma_{\hat\rho})$-valued 1-form measuring the deviation of the spin-connection ω from the Riemannian (or "Levi-Civita") spin-connection ${}^\theta\omega$ [*cf.* (4.3.6) below]:

$$K := (\omega^a{}_b - {}^\theta\omega^a{}_b) \otimes F_a{}^b,$$

$(F_a{}^b)$ denoting a local fibre basis of $\mathfrak{gl}(\Sigma_{\hat\rho})$. The components of the associated tensor field then read

$$K^{abc} = -\frac{1}{2}(\tau^{abc} + \tau^{bca} - \tau^{cab}), \tag{4.3.5}$$

$(\tau^a{}_{de} =: \tau^{abc}\eta_{bd}\eta_{ce})$ denoting the components of the tensor associated to the torsion 2-form. Finally, note that a *curvature 2-form* associated with ω may be defined as the $\mathfrak{gl}(\Sigma_{\hat\rho})$-valued 2-form

$$\Omega := \Omega^a{}_b \otimes F_a{}^b, \quad \Omega^a{}_b := \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$$

(*cf.* §2.6.2), and that the components of the Riemannian spin-connection ${}^\theta\omega$ read (*cf.*, e.g., Choquet-Bruhat 1987)

$${}^\theta\omega^{ab}{}_\mu = \theta^{b\nu}\partial_{[\nu}\theta^a{}_{\mu]} + \theta^{a\rho}\theta_{c\mu}\theta^{b\nu}\partial_{[\nu}\theta^c{}_{\rho]} + \theta^{a\nu}\partial_{[\mu}\theta^b{}_{\nu]} \equiv {}^\theta\omega^{[ab]}{}_\mu, \tag{4.3.6}$$

Latin and Greek indices being lowered or raised by η and g, respectively, or their inverses.

In the sequel, we shall need the (gauge-natural) Lie derivatives, with respect to a $\mathrm{Spin}(1,3)^e$-invariant vector field Ξ on Σ, of a spin-tetrad, a spin-connection, a spinor and its Dirac adjoint. Locally, they read

$$\pounds_\Xi\theta^a{}_\mu = \tilde\nabla_\mu\xi^\nu\theta^a{}_\nu - \check\Xi^a{}_b\theta^b{}_\mu, \tag{4.3.7a}$$

$$\pounds_\Xi\omega^a{}_b = \xi \rfloor \Omega^a{}_b + \mathcal{D}\check\Xi^a{}_b, \tag{4.3.7b}$$

$$\pounds_\Xi\psi = \xi^\nu\partial_\nu\psi + \tfrac{1}{4}\Xi^{ab}\gamma_{ab}\psi \equiv \xi^\nu\nabla_\nu\psi + \tfrac{1}{4}\check\Xi^{ab}\gamma_{ab}\psi, \tag{4.3.7c}$$

$$\pounds_\Xi\tilde\psi = \widetilde{\pounds_\Xi\psi} \equiv \xi^\nu\nabla_\nu\tilde\psi - \tfrac{1}{4}\check\Xi^{ab}\tilde\psi\gamma_{ab}, \tag{4.3.7d}$$

respectively, $(\xi^\mu, \Xi^a{}_c =: \Xi^{ab}\eta_{bc})$ denoting the components of the $\mathrm{SO}(1,3)^e$-invariant vector field on $\Sigma/\mathbb{Z}_2$ induced by Ξ [*cf.* (2.6.13′), (2.6.28) and (2.5.1)].

We are now in a position to apply the theory of conserved quantities developed in Chapter 3 to the Einstein (-Cartan) -Dirac theory. We shall do so separately for the Einstein-Cartan-Dirac case and the Einstein-Dirac one. Calculations will be "formal", unless otherwise stated, i.e. they will involve local coordinates, rather than sections, of the bundles under consideration. For the sake of simplicity, we shall nevertheless use the names of the corresponding sections. With a slight abuse of notation, we shall also use the symbols '∇' and '$\mathcal{D}$' for their formal counterparts, defined in the usual manner (*cf.*, e.g., Definition 2.1.6).

4.4 Einstein-Cartan-Dirac theory

Our main reference for the Einstein-Cartan-Dirac theory is Choquet-Bruhat (1987).

In the light of the new geometric framework developed in §4.2, the ***Einstein-Cartan***

Lagrangian can be defined as the base-preserving morphism

$$\begin{cases} \mathcal{L}_{\rm EC}\colon \Sigma_\rho \times_M J^1\Sigma_\ell \to \bigwedge^4 T^*M \\ \mathcal{L}_{\rm EC}\colon (\theta, j^1\omega) \mapsto \mathcal{L}_{\rm EC}(\theta, j^1\omega) := -\frac{1}{2\kappa}\Omega_{ab} \wedge \Sigma^{ab} \end{cases},$$

where $\kappa := 8\pi G/c^4$, $\Sigma_{ab} := e_b \lrcorner (e_a \lrcorner \Sigma)$ and Σ is the standard volume form on M locally given by $\det\|\theta\|\, \mathrm{d}x^0 \wedge \cdots \wedge \mathrm{d}x^3$. Here $\|\theta\|$ stands for the matrix of the components of θ and we have set $e_a := e_a{}^\mu \partial_\mu$, $\|e_a{}^\mu\|$ denoting the inverse of $\|\theta\|$. The ***Dirac Lagrangian*** reads instead

$$\begin{cases} \mathcal{L}_{\rm D}\colon \Sigma_\rho \times_M \Sigma_\ell \times_M J^1\Sigma_{\hat\gamma} \to \bigwedge^4 T^*M \\ \mathcal{L}_{\rm D}\colon (\theta, \omega, j^1\psi) \mapsto \mathcal{L}_{\rm D}(\theta, \omega, j^1\psi) := \left[\frac{\mathrm{i}\alpha}{2}(\tilde\psi\gamma^a\nabla_a\psi - \nabla_a\tilde\psi\gamma^a\psi) - m\tilde\psi\psi\right]\Sigma \end{cases},$$

where $\alpha := \hbar c$. According to the principle of minimal coupling, the total Lagrangian of the theory will be simply assumed to be $\mathcal{L} := \mathcal{L}_{\rm EC} + \mathcal{L}_{\rm D}$. A vertical vector field on the configuration bundle will then read

$$\Upsilon = \delta\theta^a{}_\mu \frac{\partial}{\partial\theta^a{}_\mu} + \delta\omega^a{}_{b\mu}\frac{\partial}{\partial\omega^a{}_{b\mu}} + \delta\omega^a{}_{b\mu,\nu}\frac{\partial}{\partial\omega^a{}_{b\mu,\nu}} + \delta\psi^A\frac{\partial}{\partial\psi^A} + \delta\psi^A{}_\mu\frac{\partial}{\partial\psi^A{}_\mu},$$

where $(\theta^a{}_\mu)$, $(\omega^a{}_{b\mu}, \omega^a{}_{b\mu,\nu})$ and $(\psi^A, \psi^A{}_\mu)$ denote fibre coordinates on Σ_ρ, $J^1\Sigma_\ell$ and $J^1\Sigma_{\hat\gamma}$, respectively. If we set locally

$$\delta\theta^a := \delta\theta^a{}_\mu\, \mathrm{d}x^\mu, \qquad \delta\omega^a{}_b := \delta\omega^a{}_{b\mu}\, \mathrm{d}x^\mu,$$
$$\delta\psi := \delta\psi^A\, f_A, \qquad \delta\tilde\psi := \widetilde{\delta\psi},$$

(f_A) denoting a local fibre basis of $\Sigma_{\hat\gamma}$, then the first variation formula for $\mathcal{L}$ is

$$\begin{aligned}\delta\mathcal{L} = \left(-\tfrac{1}{\kappa}G^a{}_b + T^a{}_b\right)\Sigma_a \wedge \delta\theta^b + \left(\tfrac{1}{2\kappa}\mathcal{D}\Sigma_{ab} - S_{ab}{}^c\Sigma_c\right) \wedge \delta\omega^{ab} \\ + \mathrm{d}_{\rm H}\left[-\tfrac{1}{2\kappa}\Sigma_{ab} \wedge \delta\omega^{ab} - \tfrac{\mathrm{i}\alpha}{2}(\delta\tilde\psi\gamma^a\psi - \tilde\psi\gamma^a\delta\psi)\Sigma_a\right] \\ + \alpha[\delta\tilde\psi\, e(\mathcal{L}_{\rm D}) + \tilde e(\mathcal{L}_{\rm D})\,\delta\psi]\Sigma, \quad (4.4.1)\end{aligned}$$

where G denotes the Einstein tensor associated with Ω (*cf.* §3.3.5), $\Sigma_a := e_a \lrcorner \Sigma$ and we set

$$T^a{}_b := \Theta^a{}_b - \tfrac{\alpha}{2}[\tilde\psi\, e'(\mathcal{L}_{\rm D}) + \tilde e'(\mathcal{L}_{\rm D})\,\psi]\delta^a{}_b, \qquad \Theta^a{}_b := \tfrac{\mathrm{i}\alpha}{2}(\tilde\psi\gamma^a\nabla_b\psi - \nabla_b\tilde\psi\gamma^a\psi),$$
$$e'(\mathcal{L}_{\rm D}) := \mathrm{i}\gamma^a\nabla_a\psi - m\psi, \qquad \tilde e'(\mathcal{L}_{\rm D}) := \widetilde{e'(\mathcal{L}_{\rm D})} \equiv -(\mathrm{i}\nabla_a\tilde\psi\gamma^a + m\tilde\psi),$$
$$e(\mathcal{L}_{\rm D}) := e'(\mathcal{L}_{\rm D}) - \tfrac{\mathrm{i}}{2}K^a{}_{ba}\gamma^b\psi, \qquad \tilde e(\mathcal{L}_{\rm D}) := \widetilde{e(\mathcal{L}_{\rm D})} \equiv \tilde e'(\mathcal{L}_{\rm D}) + \tfrac{\mathrm{i}}{2}K^a{}_{ba}\tilde\psi\gamma^b,$$
$$S^{abc} := -\tfrac{\mathrm{i}\alpha}{8}\tilde\psi(\gamma^{ab}\gamma^c + \gamma^c\gamma^{ab})\psi \equiv -\tfrac{\mathrm{i}\alpha}{4}\tilde\psi\gamma^{abc}\psi \equiv S^{[abc]},$$

identity (D.1.3) having been used in the last but one equality. Thus, the Einstein-Cartan-

Dirac equations are

$$G_{ab} \approx \kappa T_{ab},$$
$$\mathcal{D}\Sigma_{ab} \approx 2\kappa S_{ab}{}^{c}\Sigma_c,$$
$$\mathrm{i}\gamma^a\nabla_a\psi - m\psi - \tfrac{\mathrm{i}}{2}K^a{}_{ba}\gamma^b\psi \approx 0.$$

The first two equations are called the *first* and the *second Einstein-Cartan* (*-Dirac*) *equation*, respectively, whereas the last one is known as the (*Cartan-*) *Dirac* equation. T is the *energy-momentum tensor* of the theory (as usual: *cf.* §3.2.1), and S the *spin momentum tensor*. Now, making use of (4.3.4), the second Einstein-Cartan equation can be put into the form

$$\tau^c \wedge \Sigma_{abc} \approx 2\kappa S_{ab}{}^{c}\Sigma_c$$

or equivalently

$$\tau^{abc} \approx 2\kappa S^{abc}, \tag{4.4.2}$$

which in turn implies that the torsion tensor is completely antisymmetric on shell. Therefore, so is the contortion tensor. Indeed, from (4.3.5) and (4.4.2)

$$K^{abc} \approx -\frac{1}{2}\tau^{abc} \approx -\kappa S^{abc}.$$

Hence, the Dirac equation reduces to $e'(\mathcal{L}_{\mathrm{D}}) \approx 0$, which implies $T_{ab} \approx \Theta_{ab}$. To sum up, the above system of equations is completely equivalent to the following

$$G_{ab} \approx \kappa\Theta_{ab},$$
$$\tau^{abc} \approx 2\kappa S^{abc},$$
$$\mathrm{i}\gamma^a\nabla_a\psi - m\psi \approx 0.$$

Now, we know that we can read off the Noether current associated with $\mathcal{L}$ from its first variation (4.4.1) (*cf.*, e.g., §3.3.4). Explicitly,

$$E(\mathcal{L},\Xi) = -\xi \lrcorner \mathcal{L} - \frac{1}{2\kappa}\Sigma_{ab} \wedge \pounds_{\Xi}\omega^{ab} - \frac{\mathrm{i}\alpha}{2}(\pounds_{\Xi}\tilde{\psi}\gamma^a\psi - \tilde{\psi}\gamma^a\pounds_{\Xi}\psi)\Sigma_a.$$

After some manipulation, which makes use (*inter alia*) of (4.3.7*b*)–(4.3.7*d*) and the fact that $G^a{}_b\Sigma_a \equiv -1/2\,\Omega^{ac} \wedge (e_b \lrcorner \Sigma_{ac})$, $E(\mathcal{L},\Xi)$ can be recast as

$$E(\mathcal{L},\Xi) = \xi^b\left(-\frac{1}{\kappa}G^a{}_b + T^a{}_b\right)\Sigma_a + \check{\Xi}^{ab}\left(\frac{1}{2\kappa}\mathcal{D}\Sigma_{ab} - S_{ab}{}^{c}\Sigma_c\right) + \mathrm{d_H}\left(-\frac{1}{2\kappa}\check{\Xi}^{ab}\Sigma_{ab}\right),$$

so that the superpotential associated with $\mathcal{L}$ turns out to be

$$U(\mathcal{L},\Xi) = U(\mathcal{L}_{\mathrm{EC}},\Xi) := -\frac{1}{2\kappa}\check{\Xi}^{ab}\Sigma_{ab}, \tag{4.4.3}$$

a result which appeared in Godina *et al.* (2001) for the first time. Therefore, the Dirac Lagrangian does not seem to contribute to the total superpotential. From this fact one might mistakenly conclude that the Dirac fields do not contribute to the total conserved quantities. This conclusion would be wrong because, although the Dirac Lagrangian does not contribute *directly* to the superpotential, in order to obtain the corresponding conserved quantities, one needs to integrate the superpotential on a solution, which in turn depends on the Dirac Lagrangian via its energy-momentum tensor and the second Einstein-Cartan equation.

Note that in the case of the Kosmann lift we have[2] (*cf.* §2.6.1)

$$\check{\Xi}_{ab} = (\check{\xi}_{\mathrm{K}})_{ab} \equiv -\tilde{\nabla}_{[a}\xi_{b]}, \tag{4.4.4}$$

which, substituted into (4.4.3), gives

$$U(\mathcal{L}_{\mathrm{EC}}, \xi_{\mathrm{K}}) = \frac{1}{2\kappa}\tilde{\nabla}_a\xi_b\Sigma^{ab}, \tag{4.4.5}$$

i.e. (half of) the well-known Komar (1959) potential, in accordance with the result found by Ferraris *et al.* (1994) in a purely natural context (*cf.* §3.3.5). This is also the lift implicitly used by Godina *et al.* (2000) in the 2-spinor formalism.

Let now $({\sigma_a}^{AA'})$ denote the Infeld-van der Waerden symbols, which express the isomorphism between $\mathrm{Re}[S(M)\otimes\bar{S}(M)]$ and TM in the orthonormal basis induced by the spin-frame chosen (*cf.* §D.3), and consider the following lift[3]:

$$\xi^\mu = {e_a}^\mu {\sigma^a}_{AA'}\lambda^A\bar{\lambda}^{A'}, \qquad \check{\Xi}_{ab} = (\check{\xi}_{\mathrm{W}})_{ab} := -4{\sigma_{[a}}^{AA'}{\sigma_{b]}}^{BB'}\,\mathrm{Re}(\bar{\lambda}_{B'}\nabla_{BA'}\lambda_A), \tag{4.4.6}$$

which will be referred to as the ***Witten lift***. Then

$$U(\mathcal{L}_{\mathrm{EC}}, \xi_{\mathrm{W}}) = \mathrm{Re}\,W \equiv -\frac{2}{\kappa}\,\mathrm{Re}(\mathrm{i}\bar{\lambda}_{A'}\mathcal{D}\lambda_A\wedge\theta^{AA'}), \tag{4.4.7}$$

which is the (real) Nester-Witten 2-form (Nester 1981; Penrose & Rindler 1986). Indeed, we have[4]:

$$\begin{aligned}
\check{\Xi}_{ab}\Sigma^{ab} &= -2\bar{\lambda}_{B'}\nabla_{BA'}\lambda_A\Sigma^{ab} + \mathrm{cc}\\
&= 2\mathrm{i}^*(\bar{\lambda}_{A'}\nabla_{BB'}\lambda_A)\Sigma^{ab} + \mathrm{cc}\\
&= 2\mathrm{i}\bar{\lambda}_{A'}\nabla_b\lambda_A{}^*\Sigma^{ab} + \mathrm{cc}\\
&= -2\mathrm{i}\bar{\lambda}_{A'}\nabla_b\lambda_A\theta^a\wedge\theta^b + \mathrm{cc}\\
&= 2\mathrm{i}\bar{\lambda}_{A'}\mathcal{D}\lambda_A\wedge\theta^{AA'} + \mathrm{cc},
\end{aligned} \tag{4.4.8}$$

where we used the identities (*cf.* Penrose & Rindler 1984)

$$^*A_{ab}B^{ab} = A_{ab}{}^*B^{ab}, \qquad {}^{**}A^{ab} = -A^{ab}, \qquad {}^*A^{ABA'B'} = \mathrm{i}A^{ABB'A'}$$

for any two bivectors A^{ab} and B^{ab}. Inserting (4.4.8) into (4.4.3) gives (4.4.7), as claimed.

[2]Here, the symbol $\tilde{\nabla}_a$ stands for the operator ${e_a}^\sigma\tilde{\nabla}_\sigma$.

[3]Here, the symbol $\nabla_{AA'}$ stands for ${\sigma^a}_{AA'}{e_a}^\sigma\nabla_\sigma$.

[4]With the exception of formula (4.4.9) below, we shall suppress hereafter the Infeld-van der Waerden symbols and adopt the standard identification $a = AA'$, $b = BB'$, etc., as is customary in the current literature (*cf.* §D.3).

If we wish, it is also possible to define a ***complexified Witten lift*** as

$$\xi^\mu = e_a{}^\mu \sigma^a{}_{AA'} \lambda^A \bar{\lambda}^{A'}, \qquad \check{\Xi}_{ab} = (\check{\xi}^{\mathbb{C}}_{\mathrm{W}})_{ab} := -4\sigma_{[a}{}^{AA'} \sigma_{b]}{}^{BB'} \bar{\lambda}_{B'} \nabla_{BA'} \lambda_A. \tag{4.4.9}$$

Then, the relevant superpotential is

$$U(\mathcal{L}_{\mathrm{EC}}, \xi^{\mathbb{C}}_{\mathrm{W}}) = W := -\frac{2\mathrm{i}}{\kappa} \bar{\lambda}_{A'} \mathcal{D} \lambda_A \wedge \theta^{AA'}, \tag{4.4.10}$$

which is the (complex) Nester-Witten 2-form (Penrose & Rindler 1986; Mason & Frauendiener 1990). From the viewpoint of physical applications (proof of positivity of the Bondi or ADM mass, quasi-local definitions of momentum and angular momentum in general relativity, etc.), it is immaterial whether one uses (4.4.10) or its real part (4.4.7), as its imaginary part turns out to be $-1/\kappa \, \mathrm{d_H}(\lambda_A \bar{\lambda}_{A'} \theta^a)$, which vanishes upon integration over a closed 2-surface.
Note, though, that (4.4.10) appears to relate more directly to Penrose quasi-local 4-momentum, when suitable identifications are made (*cf.* Penrose & Rindler 1986, p. 432).

Remark 4.4.1. Note also that—modulo an inessential numerical factor—the Kosmann lift is (the real part of) the *dual* of the (complexified) Witten lift, in the sense that

$$(\check{\xi}_{\mathrm{K}})_{ab} = -\frac{1}{2} \operatorname{Re}[{}^*(\check{\xi}^{\mathbb{C}}_{\mathrm{W}})_{ab}],$$

as can be easily checked on starting from equations (4.4.4) and (4.4.9)(2), whenever, of course, $\xi^a = \lambda^A \bar{\lambda}^{A'}$.

4.4.1 Natural approach

Suppose for a moment that we deliberately neglect the gauge-natural nature of the Einstein-Cartan-Dirac theory. This means that we shall temporarily regard the Einstein-Cartan Lagrangian as a purely natural Lagrangian, i.e. a first order Lagrangian on a (purely) natural bundle. In particular, the spin-connection ω will be replaced by a *natural* linear connection Γ (*cf.* Example 1.10.17). As such, Γ is a natural object, whose components $(\Gamma^\rho{}_{\nu\mu})$ are related, because of the compatibility condition $\nabla\theta = 0$, to the components $(\omega^a{}_{b\mu})$ of ω via the familiar formula

$$\omega^a{}_{b\mu} = \theta^a{}_\rho (\partial_\mu e_b{}^\rho + \Gamma^\rho{}_{\nu\mu} e_b{}^\nu), \tag{4.4.11}$$

where antisymmetrization in $\{a, b\}$ is understood on the r.h.s. of (4.4.11) (*cf.* §2.6). Note that we *cannot* regard the Dirac Lagrangian itself as a natural Lagrangian because spinors cannot be suitably replaced by any (physically equivalent) natural objects: this is precisely why we chose a gauge-natural formulation in the first place, and why we expect to encounter some sort of restrictions now.

The local expression for the Lie derivative of Γ is given by formula (2.6.29), i.e.

$$\pounds_\xi \Gamma^\rho{}_{\nu\mu} = R^\rho{}_{\nu\sigma\mu} \xi^\sigma + \nabla_\mu \tilde{\nabla}_\nu \xi^\rho. \tag{4.4.12}$$

Thus, the Noether current is now of the form

$$E(\mathfrak{L}, \Xi) = -\xi \rfloor \mathfrak{L} - \frac{1}{2\kappa}\Sigma_{ab} \wedge \pounds_\xi \Gamma^{ab} - \frac{\mathrm{i}\alpha}{2}(\pounds_\Xi \tilde{\psi}\gamma^a\psi - \tilde{\psi}\gamma^a \pounds_\Xi \psi)\Sigma_a, \tag{4.4.13}$$

where we set

$$\pounds_\xi \Gamma^{ab} := (\pounds_\xi \Gamma^\rho{}_{\nu\mu})\theta^a{}_\rho \theta^{b\nu}\, \mathrm{d}x^\mu.$$

The important point to note here is that, although $(\Gamma^\rho{}_{\nu\mu})$ may be regarded as the components of ω in the holonomic basis, $(\pounds_\xi \Gamma^\rho{}_{\nu\mu})$ are *not*, in general, the components of $\pounds_\Xi \omega$ in that basis. Accordingly, the second term on the right-hand side of identity (4.4.13) cannot be claimed to be the most general expression for $\langle f(\mathfrak{L}_{\mathrm{EC}}), \pounds_\Xi \omega\rangle$, but naturality must indeed be assumed. In fact, if we now proceeded in the same way as before, we would then find that consistency with the second Einstein-Cartan equation requires $\check{\Xi}^{ab} = -\tilde{\nabla}^{[a}\xi^{b]}$, i.e. precisely the Kosmann lift, and thus we would recover the purely natural result.

4.5 Einstein-Dirac theory

Our main reference for the Einstein-Dirac theory is Lichnerowicz (1964). The procedure for obtaining the conserved quantities is completely analogous to the Einstein-Cartan-Dirac case; therefore, we shall limit ourselves to present the results and briefly comment on them, pointing out the possible differences. In the sequel, the symbol '$|_{K=0}$' affixed to a quantity shall mean that the latter is formally identical with the quantity denoted by the same letter in §4.4, but with all (explicit or implicit) occurrences of ω replaced by ${}^\theta\omega$.

Then, the ***Einstein-Hilbert Lagrangian*** is nothing but

$$\begin{cases} \mathfrak{L}_{\mathrm{EH}} \colon J^2\Sigma_\rho \to \bigwedge^4 T^*M \\ \mathfrak{L}_{\mathrm{EH}} \colon (j^2\theta) \mapsto \mathfrak{L}_{\mathrm{EH}}(j^2\theta) := \mathfrak{L}_{\mathrm{EC}}|_{K=0} \end{cases}, \tag{4.5.1}$$

whereas the ***Dirac Lagrangian*** is regarded here as the base-preserving morphism

$$\begin{cases} {}^\theta\mathfrak{L}_{\mathrm{D}} \colon J^1\Sigma_\rho \times_M J^1\Sigma_{\hat{\gamma}} \to \bigwedge^4 T^*M \\ {}^\theta\mathfrak{L}_{\mathrm{D}} \colon (j^1\theta, j^1\psi) \mapsto {}^\theta\mathfrak{L}_{\mathrm{D}}(j^1\theta, j^1\psi) := \mathfrak{L}_{\mathrm{D}}|_{K=0} \end{cases}.$$

Remark 4.5.1. On using (4.2.1) and exploiting the relationship between Ω and R (*cf.* §2.6.2), it is easy to see that $\mathfrak{L}_{\mathrm{EH}}$ coincides, *as an object*, with the Einstein-Hilbert Lagrangian introduced in §3.3.5, but here $\mathfrak{L}_{\mathrm{EH}}$ is regarded as a gauge-natural, not simply natural, Lagrangian.

According to the principle of minimal coupling, the total Lagrangian of the Einstein-Dirac theory will simply be ${}^\theta\mathfrak{L} := \mathfrak{L}_{\mathrm{EH}} + {}^\theta\mathfrak{L}_{\mathrm{D}}$, its variation reading

$$\begin{aligned} \delta\, {}^\theta\mathfrak{L} = \left(-\tfrac{1}{\kappa}{}^\theta G^a{}_b + {}^\theta T^a{}_b\right)\Sigma_a \wedge \delta\theta^b \\ + \mathrm{d_H}\left[-\tfrac{1}{2\kappa}\Sigma_{ab} \wedge \delta\, {}^\theta\omega^{ab} + \tfrac{1}{2}S^{ab}{}_c\Sigma_{ab} \wedge \delta\theta^c - \tfrac{\mathrm{i}\alpha}{2}(\delta\tilde{\psi}\gamma^a\psi - \tilde{\psi}\gamma^a\delta\psi)\Sigma_a\right] \\ + \alpha[\delta\tilde{\psi}\, e({}^\theta\mathfrak{L}_{\mathrm{D}}) + \tilde{e}({}^\theta\mathfrak{L}_{\mathrm{D}})\,\delta\psi]\Sigma, \end{aligned}$$

where

$$ {}^\theta T^a{}_b := T^a{}_b|_{K=0} + {}^\theta\nabla_c S^a{}_b{}^c \qquad (4.5.2) $$
$$ \equiv {}^\theta\Theta^a{}_b + \tilde{b}^a{}_b\, e({}^\theta\mathfrak{L}_\mathrm{D}) + \tilde{e}({}^\theta\mathfrak{L}_\mathrm{D})\, b^a{}_b, \qquad (4.5.3) $$
$$ {}^\theta\Theta^a{}_b := \tfrac{1}{2}(\Theta^a{}_b + \Theta_b{}^a)_{K=0} \equiv {}^\theta\Theta_b{}^a, $$
$$ b^a{}_b := \tfrac{\alpha}{4}(\gamma^a{}_b - 2\delta^a{}_b)\psi, \qquad \tilde{b}^a{}_b := \widetilde{b^a{}_b} \equiv -\tfrac{\alpha}{4}\tilde{\psi}(\gamma^a{}_b + 2\delta^a{}_b), $$
$$ e({}^\theta\mathfrak{L}_\mathrm{D}) := e'(\mathfrak{L}_\mathrm{D})|_{K=0}, \qquad \tilde{e}({}^\theta\mathfrak{L}_\mathrm{D}) := \widetilde{e({}^\theta\mathfrak{L}_\mathrm{D})} \equiv \tilde{e}'(\mathfrak{L}_\mathrm{D})|_{K=0}. $$

Thus, the Einstein-Dirac equations are

$$ {}^\theta G_{ab} \approx \kappa\, {}^\theta\Theta_{ab}, $$
$$ \mathrm{i}\gamma^a\, {}^\theta\nabla_a\psi - m\psi \approx 0. $$

Note that, although the invariance of the Dirac Lagrangian with respect to Lorentz transformations requires ${}^\theta T_{ab}$ to be symmetric *on* shell (Weinberg 1972; Choquet-Bruhat 1987), the manipulation required for going from (4.5.2) to (4.5.3) is highly non-trivial: the interested reader is referred to Lichnerowicz (1964) for an elegant proof.

Following the same procedure as before, we find that the Noether current associated with ${}^\theta\mathfrak{L}$ is

$$ E({}^\theta\mathfrak{L},\Xi) = -\xi \rfloor {}^\theta\mathfrak{L} - \tfrac{1}{2\kappa}\Sigma_{ab}\wedge \pounds_\Xi {}^\theta\omega^{ab} + \tfrac{1}{2}S^{ab}{}_c\Sigma_{ab}\wedge\pounds_\Xi\theta^c - \tfrac{\mathrm{i}\alpha}{2}(\pounds_\Xi\tilde{\psi}\gamma^a\psi - \tilde{\psi}\gamma^a\pounds_\Xi\psi)\Sigma_a $$
$$ = \xi^b\Big(-\tfrac{1}{\kappa}{}^\theta G^a{}_b + {}^\theta T^a{}_b\Big)\Sigma_a + \mathrm{d_H}\Big(-\tfrac{1}{2\kappa}\check{\Xi}^{ab}\Sigma_{ab} + \tfrac{1}{2}\xi^c S^{ab}{}_c\Sigma_{ab}\Big), $$

so that the superpotential associated with ${}^\theta\mathfrak{L}$ is recognized to be

$$ U({}^\theta\mathfrak{L},\Xi) := -\frac{1}{2\kappa}\check{\Xi}^{ab}\Sigma_{ab} + \frac{1}{2}\xi^c S^{ab}{}_c\Sigma_{ab}, \qquad (4.5.4) $$

and we note that, unlike in the Einstein-Cartan-Dirac case, the Dirac Lagrangian enters the superpotential directly, but recall that we have no second Einstein-Cartan equation here. Note also that the “vertical contribution” (i.e. all terms in $\check{\Xi}^{ab}$) coming from the Dirac Lagrangian consistently vanishes *off* shell. For the same reason, no inconsistency of the type of §4.4.1 can arise here. This fact, though, by no means disproves the gauge-naturality of the theory, which is well-motivated on both physical and mathematical grounds.

4.5.1 First order Einstein-Hilbert gravity

Remarkably, Lagrangian (4.5.1) can be split into a horizontal differential plus a ***first order* covariant *Lagrangian*** locally reading

$$ \mathfrak{L}_\mathrm{FF} := -\tfrac{1}{2\kappa}(\hat{\Omega}_{ab} - Q_{ac}\wedge Q^c{}_b)\wedge\Sigma^{ab} $$
$$ \equiv \mathfrak{L}_\mathrm{EH} + \tfrac{1}{2\kappa}\mathrm{d_H}(Q_{ab}\wedge\Sigma^{ab}), $$

where $\hat{\Omega}_{ab} := \mathrm{d}_{\mathrm{H}}\hat{\omega}_{ab} + \hat{\omega}_{ac} \wedge \hat{\omega}^{c}{}_{b}$ and $Q_{ab} := \omega_{ab} - \hat{\omega}_{ab}$, $\hat{\omega}$ being a background (non-dynamical) spin-connection. It is easy to see that $\mathfrak{L}_{\mathrm{FF}}$ is but the gauge-natural counterpart of Ferraris & Francaviglia's (1990) Lagrangian introduced in §3.3.5: in particular, $Q^{a}{}_{b} \equiv \theta^{a}{}_{\alpha} e_{b}{}^{\mu} q^{\alpha}{}_{\mu\nu}\, \mathrm{d}x^{\nu}$. Then, from (3.2.17) it follows that the superpotential associated with $\mathfrak{L}_{\mathrm{FF}}$ is

$$U(\mathfrak{L}_{\mathrm{FF}}, \Xi) := U(\mathfrak{L}_{\mathrm{EH}}, \Xi) + \frac{1}{2\kappa} \xi \rfloor (Q_{ab} \wedge \Sigma^{ab}), \tag{4.5.5}$$

where, of course, $U(\mathfrak{L}_{\mathrm{EH}}, \Xi) := -1/(2\kappa)\, \check{\Xi}^{ab} \Sigma_{ab} \equiv U(\mathfrak{L}_{\mathrm{EC}}, \Xi)$.

4.6 The indeterminacy

Both (4.4.3) and (4.5.4) reveal that, in this gauge-natural formulation of gravity coupled with Dirac fields, the superpotential is essentially indeterminate because no condition can be imposed *a priori* on the vertical part of Ξ. Therefore, we can state our main result as follows.

Theorem 4.6.1. *Any conserved charge associated with the gravitational field is intrinsically indeterminate.*

Note that, because of (3.2.17), this indeterminacy does not depend on the particular Lagrangians chosen [*cf.*, e.g., (4.5.5)]: for this reason and the functorial nature of this indeterminacy we have called it "intrinsic". This important result can be regarded either as a limit for the theory or as an additional flexibility. In any case, it cannot be overlooked.

If we look back at the examples of §3.3, we realize that, in the case of (scalar and) Yang-Mills fields, the contribution to the conserved quantities associated with a vector field ξ on M comes from its *horizontal* lift (with respect to the G-connection A), whereas the vertical contribution seems to be related to the "gauge charges", such as the electric charge in the case of electromagnetism (*cf.* §3.3.4). For the Proca fields and standard general relativity, the horizontal lift (possibly with respect to the Levi-Civita connection Γ) is no longer enough if we wish to include the contribution coming from the superpotentials, and we are thus led to consider the *natural* lift of ξ. Finally, here, as we already noted in §4.4, in order to recover the purely natural results, we should impose the *Kosmann* lift, which is canonical, but not natural, i.e. it is only one of the possible lifts of ξ onto Σ, and is not functorially induced by ξ. Remarkably, when ξ is null, another possibility is given by the (complexified) *Witten* lift, which is also related to quasi-local definitions of momentum and angular momentum in general relativity (Ludvigsen & Vickers 1983; Dougan & Mason 1991).

From a physical point of view, it might be disturbing to think, that, when the spinorial contribution is removed, the (gravitational part of the) theory should automatically revert to its purely natural counterpart, thereby reproducing the well-known (non-indeterminate) results of §3.3.5. This could mean either that some (possibly physical) justification has to be found to impose the Kosmann lift by hand[5] or, conversely, as we

[5]In this connection, it is interesting to note that, in the triad-affine formulation of the $(2+1)$-dimensional BTZ black hole solution, the Kosmann lift is precisely what is needed to recover the "one-quarter-area law" for the entropy of the black hole (Fatibene *et al.* 1999*b*).

believe, that a gauge-natural formulation is the appropriate one for gravity for the very reason that it is the most general one[6], irrespectively of the nature of the theory it is possibly coupled with.

4.7 Comparison with Giachetta *et al.*'s (1997) approach

We conclude by briefly commenting on a different approach to the same problem addressed in this chapter, developed by Sardanashvily and collaborators (*cf.* Giachetta *et al.* 1997 and references therein). We must say first of all that they consider the "metric-affine" case only, so that, strictly speaking, a comparison between the two approaches is possible only for the Einstein-Cartan-Dirac theory (although it would be relatively straightforward to extend the discussion to incorporate Einstein-Dirac gravity as well).

Indeed, even though they do not explicitly work in a gauge-natural setting, they seem to be well aware of the problems of the traditional approach mentioned in §4.1. The solution they propose is to rely on *spontaneous symmetry breaking.* Although this is admittedly a quantum phenomenon, the description of Einstein-Cartan gravity in terms of Higgs fields has some justification (*cf.*, e.g., Trautman 1980).

From our point of view and very roughly speaking, what this does is to split the gauge-naturality of the theory into its purely gauge and purely natural part, so that the gravitational contribution is still represented by a linear connection and gives rise to the usual Komar potential, whereas the indeterminate vertical contribution appears now as a further additive term, it being "decoupled". The authors, though, reject this indeterminacy and impose the Kosmann lift by hand. The final net result is $U(\mathfrak{L}, \Xi) = U(\mathfrak{L}_{\mathrm{EC}}, \xi_{\mathrm{K}})$.

It is obvious then that, in this case, their approach is, in effect, completely equivalent to the one presented here. But, if really the Kosmann lift has to be imposed by hand, then we dare favour our formulation because it is conceptually simpler and does not invoke quantum phenomena.

[6]Note that, even if we were to couple Einstein (-Cartan) gravity with "U_4-spinors" (Buchdahl 1989, 1992; Godlewski 2002), $\mathfrak{so}(1,3)$ is in some sense "maximal" since, ultimately, the superpotential of the theory must be a 2-form.

Chapter 5

Multisymplectic derivation of bi-instantaneous dynamics

> Meantime mathematicians will judge, whether in sacrificing a part of the simplicity of that geometrical conception on which the theories of LAGRANGE and POISSON are founded, a simplicity of another kind has not been introduced, which was wanting in those admirable theories.
>
> W. R. HAMILTON, *Second essay on a general method in dynamics*

The purpose of this chapter is to give a geometrical derivation of bi-instantaneous dynamics in the sense of Hayward (1993). This will be done by appealing to the multisymplectic (Hamiltonian) formalism.

5.1 Motivation

The standard 3+1 Hamiltonian analysis of the gravitational field, or "ADM formalism", due to Arnowitt *et al.* (1959, 1960, 1962) formulates gravitational dynamics in terms of the evolution of spacelike hypersurfaces. In many cases, however, particularly in problems where gravitational radiation is important, it is desirable to consider a foliation by characteristic or null hypersurfaces. The geometry of a space-time foliated by null hypersurfaces is rather awkward to describe owing to the absence of a natural rigging vector and the degeneracy of the metric on a null surface. However, if one further decomposes the null surfaces into families of spacelike 2-surfaces, one obtains a special case of the 2+2 formalism in which no such degeneracies occur. The 2+2 formalism decomposes space-time into two families of spacelike 2-surfaces. We can view this as a constructive procedure in which an initial 2-dimensional submanifold is chosen in a bare manifold together with two vector fields which transvect the submanifold everywhere. The two vector fields can then be used to drag the initial 2-surface out into two foliations of 3-surfaces. The character of these 3-surfaces will depend in turn on the character of the two vector fields. Since, once a metric is introduced, the two vector fields may each separately be null, timelike or spacelike, then this gives rise to six different types of decomposition. The two most important cases are double-null foliations and null-timelike foliations. An elegant way of describing this decomposition is to introduce a manifestly covariant formalism in which one uses projection operators and Lie derivatives in the normal directions. This approach gives rise to the covariant 2+2 formalism of d'Inverno and collaborators (d'Inverno & Stachel 1978; d'Inverno & Smallwood 1980; Smallwood 1983).

Since in general relativity space-time is represented by a (Lorentzian) 4-dimensional manifold, it follows that, geometrically, the 2+2 approach lies exactly in between the standard 3+1 and the multisymplectic formalism presented in the following section.

5.2 Multisymplectic formulation of a field theory

We shall now recall the basic ingredients of the multisymplectic (Hamiltonian) formalism. This section closely follows Gotay *et al.* (1998), §§2B and 3A–B, to which the reader is referred for more detail and an extensive bibliography.

First of all, we need to introduce the field-theoretic analogue of the cotangent bundle. Let then B denote the configuration bundle of a *first order*[1] field theory. As in Chapter 3, we could develop our formalism for a generic fibre bundle B over an m-dimensional manifold M, but hereafter we shall restrict attention to the case in which B is a gauge-natural bundle P_λ associated with some principal bundle $P(M,G)$. We *define* the ***dual jet bundle*** $J^1P^*_\lambda$ to be the *vector* bundle over P_λ whose fibre at $y \in (P_\lambda)_x$ is the set of *affine* maps from $J^1_yP_\lambda$ to $\bigwedge^m T^*_xM$. A section of $J^1P^*_\lambda$ is therefore an affine bundle map of J^1P_λ to $\bigwedge^m T^*M$ covering the projection $\pi\colon P_\lambda \to M$. Fibre coordinates on $J^1P^*_\lambda$ are $(p, p_{\mathfrak{a}}{}^\mu)$, which correspond to the affine map given in coordinates by

$$y^{\mathfrak{a}}{}_\mu \mapsto (p + p_{\mathfrak{a}}{}^\mu y^{\mathfrak{a}}{}_\mu)\, \mathrm{d}s. \tag{5.2.1}$$

Analogous to the canonical 1- and 2-forms on a cotangent bundle, there are canonical forms on $J^1P^*_\lambda$. To define these, another description of $J^1P^*_\lambda$ will be convenient. Namely, let $\Lambda_\lambda := \bigwedge^m T^*P_\lambda$ denote the bundle of m-forms on P_λ, with fibre over $y \in P_\lambda$ denoted by $(\Lambda_\lambda)_y$ and with projection $\pi_\Lambda\colon \Lambda_\lambda \to P_\lambda$. Let Z_λ be the subbundle of Λ_λ whose fibre is given by

$$(Z_\lambda)_y := \{\, z \in (\Lambda_\lambda)_y \mid v' \lrcorner\, v \lrcorner\, z = 0 \ \forall v, v' \in V_yP_\lambda \,\}. \tag{5.2.2}$$

Elements of Z_λ can be be written uniquely as

$$z = p\, \mathrm{d}s + p_{\mathfrak{a}}{}^\mu\, \mathrm{d}y^{\mathfrak{a}} \wedge \mathrm{d}s_\mu. \tag{5.2.3}$$

Hence, fibre coordinates for Z_λ are also $(p, p_{\mathfrak{a}}{}^\mu)$ and we note that $Z_\lambda \cong T^*P_\lambda \wedge (\bigwedge^{m-1} T^*M)$. Z_λ, which is clearly a gauge-natural bundle over M, is called the (***homogeneous***) ***Legendre bundle*** (*cf.*, e.g., Giachetta *et al.* 1997).

Equating the coordinates $(x^\lambda, y^{\mathfrak{a}}, p, p_{\mathfrak{b}}{}^\mu)$ of Z_λ and of $J^1P^*_\lambda$ defines a vector bundle isomorphism

$$\Phi\colon Z_\lambda \to J^1P^*_\lambda. \tag{5.2.4}$$

[1]The formalism presented in this section can be generalized to the higher order case, but we shall not need to do so here. The interested reader is referred to Ferraris & Francaviglia (1983*b*), Gotay (1991*a*) and references therein.

Intrinsically, Φ is defined by

$$\Phi(z) \circ \dot{y} := \dot{y}^* z \in \bigwedge^m T^*_x M, \tag{5.2.5}$$

where $z \in (Z_\lambda)_y$, $\dot{y} \in J^1_y P_\lambda$ and $x = \pi(y)$. To see this, note that, if $\dot{y}$ has fibre coordinates $(y^{\mathfrak{a}}{}_\mu)$, then

$$\dot{y}^* \mathrm{d}x^\mu = \mathrm{d}x^\mu \quad \text{and} \quad \dot{y}^* \mathrm{d}y^{\mathfrak{a}} = y^{\mathfrak{a}}{}_\mu \, \mathrm{d}x^\mu \tag{5.2.6}$$

and so

$$\dot{y}^*(p \, \mathrm{d}s + p_{\mathfrak{a}}{}^\mu \, \mathrm{d}y^{\mathfrak{a}} \wedge \mathrm{d}s_\mu) = (p + p_{\mathfrak{a}}{}^\mu y^{\mathfrak{a}}{}_\mu) \, \mathrm{d}s, \tag{5.2.7}$$

where we used (1.1.2a).

We shall now construct canonical forms on Z_λ and then use the isomorphism between $J^1 P^*_\lambda$ and Z_λ to transfer these to $J^1 P^*_\lambda$. We first define the *canonical m-form* Θ_Λ on Λ_λ by

$$\begin{aligned} \underset{m}{v} \lrcorner \cdots \lrcorner \underset{1}{v} \lrcorner \Theta_\Lambda(z) &= T\pi_\Lambda \circ \underset{m}{v} \lrcorner \cdots \lrcorner T\pi_\Lambda \circ \underset{1}{v} \lrcorner z \\ &= \left(\pi^*_\Lambda z\right)(\underset{1}{v}, \ldots, \underset{m}{v}), \end{aligned} \tag{5.2.8}$$

where $z \in \Lambda_\lambda$ and $\underset{1}{v}, \ldots, \underset{m}{v} \in T_z \Lambda_\lambda$. Define the *canonical* $(m+1)$*-form* Ω_Λ on Λ_λ by

$$\Omega_\Lambda := -\mathrm{d}\Theta_\Lambda. \tag{5.2.9}$$

Note that, if $m = 1$, then $\Lambda_\lambda = T^* P_\lambda$ and Θ_Λ is the standard canonical 1-form. If $i \colon Z_\lambda \to \Lambda_\lambda$ denotes the inclusion, the ***canonical*** m***-form*** Θ on Z_λ is defined by

$$\Theta := i^* \Theta_\Lambda \tag{5.2.10}$$

and the ***canonical*** $(m+1)$***-form*** Ω on Z_λ is defined by

$$\Omega := -\mathrm{d}\Theta \equiv i^* \Omega_\Lambda. \tag{5.2.11}$$

The pair (Z_λ, Ω) is called ***multiphase space*** or ***covariant phase space***. It is an example of a *multisymplectic manifold*, i.e. a manifold equipped with a closed non-degenerate $(m+1)$-form.

From (5.2.3), (5.2.8), (5.2.9), (5.2.10) and (5.2.11), one finds that the coordinate expression for Θ is

$$\Theta = p_{\mathfrak{a}}{}^\mu \, \mathrm{d}y^{\mathfrak{a}} \wedge \mathrm{d}s_\mu + p \, \mathrm{d}s, \tag{5.2.12}$$

and so

$$\Omega = \mathrm{d}y^{\mathfrak{a}} \wedge \mathrm{d}p_{\mathfrak{a}}{}^\mu \wedge \mathrm{d}s_\mu - \mathrm{d}p \wedge \mathrm{d}s. \tag{5.2.13}$$

Now, we shall construct the ***covariant Legendre transform*** for a first order Lagrangian $\mathfrak{L} \colon J^1 P_\lambda \to \bigwedge^m T^* M$. This is a fibre-preserving morphism

$$\mathbb{F}\mathfrak{L} \colon J^1 P_\lambda \to Z_\lambda \cong J^1 P^*_\lambda$$

over P_λ, which has the coordinate expressions

$$p_{\mathfrak{a}}{}^\mu = f_{\mathfrak{a}}{}^\mu, \quad p = L - f_{\mathfrak{a}}{}^\mu y^{\mathfrak{a}}{}_\mu \tag{5.2.14}$$

for the ***multimomenta*** $(p_{\mathfrak{a}}{}^{\mu})$ and the ***covariant Hamiltonian*** $H := -p$, the $f_{\mathfrak{a}}{}^{\mu}$'s being defined by (3.1.14). An intrinsic definition follows.

If $\dot{y} \in J^1_y P_\lambda$, then $\mathbb{F}\mathfrak{L}(\gamma)$ is to be an affine map from $J^1_y P_\lambda$ to $\bigwedge^m T^*_x M$ where $y \in (P_\lambda)_x$. Define $\mathbb{F}\mathfrak{L}(\dot{y})$ to be the first order vertical Taylor approximation to $\mathfrak{L}$, i.e.:

$$\mathbb{F}\mathfrak{L}(\dot{y}) \circ \dot{y}' := \mathfrak{L}(\dot{y}) + \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{L}\Big(\dot{y} + t(\dot{y}' - \dot{y})\Big)\bigg|_{t=0}, \tag{5.2.15}$$

where $\dot{y}' \in J^1_y P_\lambda$. To derive the coordinate expressions (5.2.14), suppose that, locally, $\dot{y} \equiv (x^\lambda, y^{\mathfrak{a}}, y^{\mathfrak{a}}{}_{\mu})$ and $\dot{y}' \equiv (x^\lambda, y^{\mathfrak{a}}, y'^{\mathfrak{a}}{}_{\mu})$. Then the right-hand side of (5.2.15) reads

$$\Big(L(\dot{y}) + f_{\mathfrak{a}}{}^{\mu}(y'^{\mathfrak{a}}{}_{\mu} - y^{\mathfrak{a}}{}_{\mu})\Big)\,\mathrm{d}s,$$

which is an affine function of $y'^{\mathfrak{a}}{}_{\mu}$ with linear and constant pieces given by the first and second equation of (5.2.14), respectively. Hence (5.2.14) is indeed the coordinate description of $\mathbb{F}\mathfrak{L}$. Then, the so-called (***Poincaré-***) ***Cartan form*** is the m-form $\Theta_{\mathfrak{L}}$ on $J^1 P_\lambda$ intrinsically defined as

$$\Theta_{\mathfrak{L}} := (\mathbb{F}\mathfrak{L})^*\Theta, \tag{5.2.16}$$

where Θ is the canonical m-form on Z_λ. Indeed, (5.2.12) and (5.2.14) yield

$$\begin{aligned}\Theta_{\mathfrak{L}} &= f_{\mathfrak{a}}{}^{\mu}\,\mathrm{d}y^{\mathfrak{a}} \wedge \mathrm{d}s_\mu + (L - f_{\mathfrak{a}}{}^{\mu} y^{\mathfrak{a}}{}_{\mu})\,\mathrm{d}s \\ &\equiv \mathfrak{L} + f_{\mathfrak{a}}{}^{\mu}\,\mathrm{d}_{\mathrm{V}} y^{\mathfrak{a}} \wedge \mathrm{d}s_\mu,\end{aligned}$$

where the second equality follows from (1.8.21). If $f_{\mathfrak{L}}$ denotes the contact 1-form uniquely associated with the Poincaré-Cartan morphism $f(\mathfrak{L})$, i.e. locally $f_{\mathfrak{L}} \equiv f_{\mathfrak{a}}{}^{\mu}\,\vartheta^{\mathfrak{a}} \wedge \mathrm{d}s_\mu \equiv f_{\mathfrak{a}}{}^{\mu}\,\mathrm{d}_{\mathrm{V}} y^{\mathfrak{a}} \wedge \mathrm{d}s_\mu$, then of course

$$\Theta_{\mathfrak{L}} = \mathfrak{L} + f_{\mathfrak{L}},$$

a formula which also holds in the k-th order case, although, in general, $\Theta_{\mathfrak{L}}$ will be global but not unique (*cf.* §3.1.2). It is now easy to see that the Noether current $E(\mathfrak{L}, \Xi)$ defined in §3.2 can be written as

$$E(\mathfrak{L}, \Xi) = -h(J^1\Xi \,\lrcorner\, \Theta_{\mathfrak{L}})$$

[*cf.* (3.2.10) and (2.1.12)]: again, this formula can be generalized to the k-th order case by replacing $J^1\Xi$ with $J^{2k-1}\Xi$. It is not surprising, then, that the so-called (***special***) ***covariant momentum map*** $\mathcal{J} \colon Z_\lambda \to \mathfrak{X}^*_G(P) \otimes \bigwedge^{m-1} T^* Z_\lambda$, defined by

$$\langle \mathcal{J}, \Xi \rangle = \Xi_Z \,\lrcorner\, \Theta$$

for all G-invariant vector field $\Xi \in \mathfrak{X}_G(P)$ on P, encodes all the essential information about the conserved quantities associated with the given (first order) field theory. (Here, Ξ_Z denotes gauge-natural lift of Ξ onto Z_λ.)

Remark 5.2.1. Our definition of a covariant momentum map differs slightly from the original one due to Gotay *et al.* (1998), but is more suitable for our gauge-natural setting. Indeed, in such a setting one has automatically defined a Lie group G and vector fields on (P_λ and) Z_λ functorially induced by G-invariant vector fields on P.

Finally, in analogy with classical mechanics, we expect the dynamics of the theory to be encoded in the canonical $(m+1)$-form Ω. Indeed, if we define

$$\Omega_{\mathfrak{L}} := (\mathbb{F}\mathfrak{L})^*\Omega \equiv -\mathrm{d}\Theta_{\mathfrak{L}},$$

then we have the following result.

Theorem 5.2.2. *Let σ be a critical section of P_λ. Then,*

$$(j^1\sigma)^*(\Xi \lrcorner \Omega_{\mathfrak{L}}) = 0 \tag{5.2.17}$$

for any vector field Ξ on J^1P_λ.

Proof. To prove the theorem, first observe that any tangent vector on J^1P_λ can be decomposed into a component tangent to the image of $j^1\sigma$ and a vertical vector on $J^1P_\lambda \to M$. Similarly, any vertical vector on $J^1P_\lambda \to M$ can be decomposed into a jet extension of some vertical vector on P_λ and a vertical vector $J^1P_\lambda \to P_\lambda$. Assume then that Ξ is tangent to the graph of $j^1\sigma$ in J^1P_λ; that is, $\Xi = Tj^1\sigma \circ \xi$ for some vector field ξ on M. Then,

$$\begin{aligned}(j^1\sigma)^*(\Xi \lrcorner \Omega_{\mathfrak{L}}) &= (j^1\sigma)^*\Big((Tj^1\sigma \circ \xi) \lrcorner \Omega_{\mathfrak{L}}\Big)\\ &= \xi \lrcorner (j^1\sigma)^*\Omega_{\mathfrak{L}},\end{aligned}$$

which vanishes since $(j^1\sigma)^*\Omega_{\mathfrak{L}}$ is an $(m+1)$-form on the m-dimensional manifold M. Assume now that Ξ is a vertical vector field on $J^1P_\lambda \to P_\lambda$. Then, locally,

$$\Xi = \Xi^{\mathfrak{a}}{}_\mu \partial_{\mathfrak{a}}{}^\mu.$$

A calculation using the coordinate expression of $\Omega_{\mathfrak{L}}$, i.e.

$$\Omega_{\mathfrak{L}} = \mathrm{d}y^{\mathfrak{a}} \wedge \mathrm{d}f_{\mathfrak{a}}{}^\mu \wedge (\mathrm{d}s_\mu - f_{\mathfrak{a}}{}^\mu y^{\mathfrak{a}}{}_\mu) \wedge \mathrm{d}s, \tag{5.2.18}$$

shows that

$$\Xi \lrcorner \Omega_{\mathfrak{L}} = -\Xi^{\mathfrak{b}}{}_\nu \frac{\partial^2 L}{\partial y^{\mathfrak{a}}{}_\mu \partial y^{\mathfrak{b}}{}_\nu}(\mathrm{d}y^{\mathfrak{a}} \wedge \mathrm{d}s_\mu - y^{\mathfrak{a}}{}_\mu\, \mathrm{d}s),$$

which clearly vanishes when pulled back by the jet of a section of P_λ because of (1.7.1) and (1.1.2a). Finally, one readily computes in coordinates that, along $j^1\sigma$,

$$(j^1\sigma)^*(J^1\Upsilon \lrcorner \Omega_{\mathfrak{L}}) = -\langle e(\mathfrak{L}) \circ j^2\sigma, \Upsilon\rangle \tag{5.2.19}$$

for all vertical vector fields Υ on P_λ. Thus, (5.2.17) implies (3.1.16). On the other hand, (5.2.19) combined with the above remarks on decompositions of vector fields shows that (3.1.16) implies (5.2.17). □

5.3 Transition from the multisymplectic to the instantaneous formalism

In this section we shall summarize the procedure devised by Gotay (1991b) for the transition from the multisymplectic to the *instantaneous* (i.e. ADM) Hamiltonian formalism (see also Giachetta *et al.* 1997; Gotay *et al.* 1999).

Let P_λ be a gauge-natural bundle over an m-dimensional manifold M and let $\Sigma \subset M$ be a compact $(m-1)$-dimensional *Cauchy* (i.e. non-characteristic) hypersurface without boundary for a (first order) gauge-natural field theory on P_λ. Let $\mathcal{P}_\lambda|_\Sigma$ be the space of all (smooth) sections of P_λ restricted to Σ. Note that, when completed in the appropriate Sobolev topology, the space $\mathcal{P}_\lambda|_\Sigma$ becomes a smooth infinite-dimensional manifold. Also, in the sequel we shall always assume to have chosen coordinates $(x^\mu)_{\mu=0}^{m-1} = (x^0, x^i)_{i=1}^{m-1}$ on M *adapted* to Σ, in the sense that Σ is locally given by $x^0 = 0$.

The tangent space $T_\sigma\mathcal{P}_\lambda|_\Sigma$ at a section $\sigma\colon \Sigma \to P_\lambda|_\Sigma$ is defined as the set of sections Υ of the vertical bundle $V\mathcal{P}_\lambda|_\Sigma \to \Sigma$ which cover σ, i.e. such that $\nu_{P_\lambda} \circ \Upsilon = \sigma$ (*cf.* §1.2). Similarly, the cotangent space $T^*_\sigma\mathcal{P}_\lambda|_\Sigma$ consists of the section of the bundle $V^*\Sigma \otimes_{P_\lambda|_\Sigma} \bigwedge^{m-1}T^*\Sigma \to \Sigma$ which cover σ. The natural pairing of an element $\alpha \in T^*_\sigma\mathcal{P}_\lambda|_\Sigma$ with an element $v \in T_\sigma\mathcal{P}_\lambda|_\Sigma$ is then given by integration:

$$\langle \alpha, v\rangle := \int_\Sigma \langle \alpha(x), v(x)\rangle, \quad x \in \Sigma.$$

Now, let $Z_\lambda|_\Sigma$ denote the restriction of the Legendre bundle Z_λ to Σ. Of course, the space $\mathcal{Z}_\lambda|_\Sigma$ of its sections is endowed with the induced fibration $\mathcal{Z}_\lambda|_\Sigma \to \mathcal{P}_\lambda|_\Sigma$. Let $\sigma \in \mathcal{Z}_\lambda|_\Sigma$ and $u, v \in T_\sigma\mathcal{Z}_\lambda|_\Sigma$. We can then define the canonical 1- and 2-form on $\mathcal{Z}_\lambda|_\Sigma$ by

$$u \lrcorner \Theta_\Sigma = \int_\Sigma \sigma^*(u \lrcorner \Theta)$$

and

$$v \lrcorner u \lrcorner \Omega_\Sigma = \int_\Sigma \sigma^*(v \lrcorner u \lrcorner \Omega) \equiv -\mathrm{d}\Theta_\Sigma,$$

respectively, Θ and Ω being the canonical forms on Z_λ defined in the previous section. In general, though, Ω_Σ fails to be symplectic because of a non-trivial kernel. Indeed, $\ker\Omega_\Sigma \equiv \operatorname{span}\{\partial/\partial p, \partial/\partial p_{\mathfrak{a}}{}^i\}$ by inspection. For this reason, Ω_Σ is called *presymplectic*. To overcome this difficulty, we define a vector bundle morphism $R_\Sigma\colon \mathcal{Z}_\lambda|_\Sigma \to T^*\mathcal{P}_\lambda|_\Sigma$ by

$$\langle R_\Sigma(\sigma), v\rangle := \int_\Sigma \Big(\pi_Z \circ \sigma\Big)^*(v \lrcorner \sigma), \tag{5.3.1}$$

$\pi_Z\colon Z_\lambda \to P_\lambda$ being the canonical projection and v an element of $T_{(\pi_Z\circ\sigma)}\mathcal{P}_\lambda|_\Sigma$. In adapted coordinates, $\sigma \in \mathcal{Z}_\lambda|_\Sigma$ takes the form

$$\sigma = p_{\mathfrak{a}}{}^\mu \, \mathrm{d}y^{\mathfrak{a}} \wedge \mathrm{d}s_\mu + p \, \mathrm{d}s, \tag{5.3.2}$$

and so we may locally write

$$R_\Sigma(\sigma) = p_{\mathfrak{a}}{}^0 \, \mathrm{d}y^{\mathfrak{a}} \otimes \mathrm{d}s_0.$$

R_Σ is obviously a surjective submersion with

$$\ker R_\Sigma = \{\, \sigma \in \mathcal{Z}_\lambda|_\Sigma : \sigma = p_{\mathfrak{a}}{}^i \, \mathrm{d}y^{\mathfrak{a}} \otimes \mathrm{d}s_i + p \, \mathrm{d}s, \; i = 1, \ldots, m-1 \,\}.$$

Proposition 5.3.1 (Gotay 1991*b*). *The quotient map* $\mathcal{Z}_\lambda|_\Sigma / \ker R_\Sigma = \mathcal{Z}_\lambda|_\Sigma / \ker\Omega_\Sigma \to T^*\mathcal{P}_\lambda|_\Sigma$ *induced by* R_Σ *is a* symplectic diffeomorphism, *i.e.* $\mathcal{Z}_\lambda|_\Sigma / \ker\Omega_\Sigma$ *is canonically isomorphic to* $T^*\mathcal{P}_\lambda|_\Sigma$, *and the inherited symplectic form on the former is isomorphic to* *the canonical one on the latter.*

Explicitly,

$$\Theta_\Sigma = R^*\theta_\Sigma,$$
$$\Omega_\Sigma = R^*\omega_\Sigma,$$

where θ_Σ and ω_Σ are the canonical 1- and 2-form on $T^*\mathcal{P}_\lambda|_\Sigma$, respectively, defined in the usual manner, i.e.

$$\langle\theta_\Sigma(\psi,\pi), v\rangle = \textstyle\int_\Sigma \pi(T\pi_{\mathcal{P}_\lambda}\circ v),$$
$$\omega_\Sigma := -\mathrm{d}\theta_\Sigma,$$

(ψ,π) denoting a point in $T^*\mathcal{P}_\lambda|_\Sigma$, v an element of $T_{(\psi,\pi)}T^*\mathcal{P}_\lambda|_\Sigma$, and $\pi_{\mathcal{P}_\lambda}\colon T^*\mathcal{P}_\lambda|_\Sigma \to \mathcal{P}_\lambda|_\Sigma$ being the cotangent bundle projection. Locally,

$$\theta_\Sigma(\psi,\pi) = \textstyle\int_\Sigma \pi_{\mathfrak{a}}\,\mathrm{d}\psi^{\mathfrak{a}}\otimes\mathrm{d}s_0,$$
$$\omega_\Sigma(\psi,\pi) = \textstyle\int_\Sigma \mathrm{d}\psi^{\mathfrak{a}}\wedge\mathrm{d}\pi_{\mathfrak{a}}\otimes\mathrm{d}s_0.$$

Now, to discuss dynamics, namely how fields evolve in time, we need to introduce the concept of a "slicing", which is a way to define a global notion of "time". It is important to note that in this concept are encoded both the idea of a *foliation* of M by means of a one-parameter family of lower dimensional hypersurfaces and that of a *fibration* through a parameter time t.

Definition 5.3.2. A ***slicing*** of an m-dimensional manifold M consists of a "reference" $(m-1)$-dimensional Cauchy hypersurface Σ and a diffeomorphism $s_M\colon \Sigma\times\mathbb{R}\to M$.

For $t\in\mathbb{R}$, we shall write Σ_t for $s_M(\Sigma\times\{t\})$. Also, we shall usually denote by ξ the *generator* of s_M, i.e. $\xi := (s_M)_*(\partial/\partial t)$, which is then said to be an ***infinitesimal slicing*** of M.

Given a bundle B over M and a slicing s_M of M, a ***compatible slicing*** of B is a bundle B_Σ over Σ and a bundle isomorphism $s_B\colon B_\Sigma\times\mathbb{R}\to B$ such that the diagram

$$\begin{array}{ccc} B_\Sigma\times\mathbb{R} & \xrightarrow{s_B} & B \\ \downarrow & & \downarrow \\ \Sigma\times\mathbb{R} & \xrightarrow[s_M]{} & M \end{array}$$

commutes, the vertical arrows denoting the appropriate bundle projections. We shall write B_t for $s_B(B_\Sigma\times\{t\})$. Also, the *generator* Ξ of s_B is given by $\Xi := (s_B)_*(\partial/\partial t)$, which is then a ***compatible infinitesimal slicing*** of B. It is easy two see that Ξ is compatible with ξ iff it projects onto ξ. Since we shall deal only with gauge-natural bundles P_λ associated with G-principal bundles P and shall consider only gauge-natural lifts Ξ_λ of G-invariant vector fields Ξ on P as infinitesimal slicings of P_λ, these will always be compatible with the infinitesimal slicings ξ of M on which they happen to project. In other words, we shall call Ξ_λ an infinitesimal slicing of P_λ if its projection ξ on M is an infinitesimal slicing of M.

Now, fix an infinitesimal slicing Ξ_λ of the configuration bundle P_λ of a first order gauge-natural field theory described by a Lagrangian $\mathcal{L}\colon J^1P_\lambda \to \bigwedge^m T^*M$. The corresponding "slice" $(P_\lambda)_t$ of P_λ is, of course, a gauge-natural bundle over the "slice" Σ_t of M corresponding to the projection ξ of Ξ_λ on M. Let then $\tilde{\mathcal{L}}_\Xi y_t\colon (J^1P_\lambda)_t \to (VP_\lambda)_t$ denote the restriction of the formal generalized Lie derivative to $(P_\lambda)_t$ (*cf.* Definition 2.1.6). Explicitly,

$$\tilde{\mathcal{L}}_\Xi y_t \circ j^1_x\sigma \equiv \tilde{\mathcal{L}}_\Xi \sigma(x)$$

for all $\sigma \in \mathcal{P}_\lambda$ and $x \in \Sigma_t$. For future convenience, set

$$\psi := \sigma|_{\Sigma_t}, \qquad \dot{\psi} := \tilde{\mathcal{L}}_\Xi y_t \circ j^1\sigma.$$

for all $\sigma \in \mathcal{P}_\lambda$. Hence define a bundle map $\beta_\Xi\colon (J^1P_\lambda)_t \to J^1(P_\lambda)_t \times (VP_\lambda)_t$ over $(P_\lambda)_t$ by

$$\beta_\Xi(j^1_x\sigma) = (j^1_x\psi, \dot{\psi}(x)) \tag{5.3.3}$$

for all $\sigma \in \mathcal{P}_\lambda$ and $x \in \Sigma_t$. In coordinates adapted to Σ_t, (5.3.3) reads

$$\beta_\Xi\Big((x^\mu, y^{\mathfrak{a}}, y^{\mathfrak{a}}{}_\nu) \circ j^1_x\sigma\Big) = \Big((x^\mu, y^{\mathfrak{a}}, y^{\mathfrak{a}}{}_i) \circ j^1_x\psi, \dot{y}^{\mathfrak{a}} \circ \dot{\psi}(x)\Big).$$

Furthermore, if the coordinates on P_λ are arranged so that $\partial/\partial x^0|_{(P_\lambda)_t} = \xi$, then of course $\dot{y}^{\mathfrak{a}} = y^{\mathfrak{a}}{}_0$, which is to say that, if ξ is transverse to Σ_t, then β_Ξ is a bundle isomorphism. In this case, β_Ξ is called the ***jet decomposition map***, and its inverse the ***jet reconstruction map***. Clearly, both maps can be extended to maps of sections. Indeed, from (5.3.3) it follows that

$$\beta_\Xi(j^1\sigma \circ i_\Sigma) = (j^1\psi, \dot{\psi}), \tag{5.3.4}$$

where $i_\Sigma\colon \Sigma_t \to M$ is the inclusion. In fact, in (5.3.4) $j^1\psi$ is completely determined by ψ. On the other hand, $\dot{\psi}$ is a section of $V(P_\lambda)_t$ covering ψ, and so defines an element of $T_\psi(\mathcal{P}_\lambda)_t$, $(\mathcal{P}_\lambda)_t$ denoting the space of sections of $(P_\lambda)_t$. Therefore, β_Ξ induces an isomorphism of $j^1(\mathcal{P}_\lambda)_t$ with $T(\mathcal{P}_\lambda)_t$, $j^1(\mathcal{P}_\lambda)_t$ denoting the collection of restrictions of holonomic sections of J^1P_λ to Σ_t.

We are now in a position to define the ***instantaneous Lagrangian*** $L_\Xi\colon T(\mathcal{P}_\lambda)_t \to \mathbb{R}$ by

$$L_\Xi(\psi, \dot{\psi}) \equiv \int_{\Sigma_t} \mathcal{L}_\Xi(\psi, \dot{\psi}) = \int_{\Sigma_t} i^*_\Sigma\Big(\xi \lrcorner \mathcal{L}(j^1\sigma)\Big),$$

for $(\psi, \dot{\psi}) \in T(\mathcal{P}_\lambda)_t$, where $j^1\sigma \circ i_\Sigma$ is the reconstruction of $(j^1\psi, \dot{\psi})$. In coordinates adapted to Σ_t, it reads

$$L_\Xi(\psi, \dot{\psi}) = \int_{\Sigma_t} L(j^1\psi, \dot{\psi})\xi^0 \,\mathrm{d}s_0.$$

The instantaneous Lagrangian L_Ξ has an ***instantaneous Legendre transform***

$$\begin{cases} \mathbb{F}L_\Xi\colon T(\mathcal{P}_\lambda)_t \to T^*(\mathcal{P}_\lambda)_t \\ \mathbb{F}L_\Xi\colon (\psi, \dot{\psi}) \mapsto (\psi, \pi) \end{cases},$$

defined in the usual way as the fibre derivative of L_Ξ. In adapted coordinates,

$$\pi = \pi_{\mathfrak{a}}\, \mathrm{d}y^{\mathfrak{a}} \otimes \mathrm{d}s_0,$$

where

$$\pi_{\mathfrak{a}} \circ (\psi, \dot{\psi}) \equiv \frac{\partial \mathcal{L}_\Xi}{\partial \dot{y}^{\mathfrak{a}}} \circ (\psi, \dot{\psi}).$$

We call

$$\mathbf{\Phi}_t := \mathbb{F}L_\Xi(T(\mathcal{P}_\lambda)_t) \subseteq T^*(\mathcal{P}_\lambda)_t$$

the ***instantaneous primary constraint set***. Constraint analysis is one of the most interesting (and difficult) aspects of Hamiltonian dynamics, and is beyond the scope of the present discussion[2]. On $\mathbf{\Phi}_t$, we can define the ***instantaneous Hamiltonian*** H_Ξ by

$$H_\Xi(\psi, \pi) = \langle \pi, \dot{\psi} \rangle - L_\Xi(\psi, \dot{\psi}).$$

In coordinates,

$$H_\Xi(\psi, \pi) \equiv \int_{\Sigma_t} \mathcal{H}_\Xi(\psi, \pi) = \int_{\Sigma_t} \Big(\pi_{\mathfrak{a}} \dot{\psi}^{\mathfrak{a}} - L(j^1\psi, \dot{\psi})\xi^0 \Big)\, \mathrm{d}s_0.$$

We have

$$H_\Xi(\psi, \pi) = - \int_{\Sigma_t} \sigma^*(\Xi_Z \lrcorner \Theta) \tag{5.3.5}$$

for any $\sigma \in R_\Sigma^{-1}(\psi, \pi) \cap \mathbb{F}\mathcal{L}(j^1(\mathcal{P}_\lambda)_t)$ and any gauge-natural lift Ξ_Z of Ξ onto Z_λ, where $R_\Sigma := R_{\Sigma_t}$ denotes the symplectic reduction morphism defined by (5.3.1). We already know that (5.3.5) must be true because it is so for its Lagrangian counterpart (*cf.* Remark 3.2.15 and §5.2). Explicitly,

$$\begin{aligned} -\textstyle\int_{\Sigma_t} \sigma^*(\Xi_Z \lrcorner \Theta) &= \textstyle\int_{\Sigma_t} [p_{\mathfrak{a}}{}^0(\xi^\mu \partial_\mu \sigma^{\mathfrak{a}} - \Xi^{\mathfrak{a}}) - (p_{\mathfrak{a}}{}^\mu \partial_\mu \sigma^{\mathfrak{a}} + p)\xi^0]\, \mathrm{d}s_0 \\ &= \textstyle\int_{\Sigma_t} [p_{\mathfrak{a}}{}^0 \pounds_\Xi \sigma^{\mathfrak{a}} - L(j^1\sigma)\xi^0]\, \mathrm{d}s_0 \\ &= \textstyle\int_{\Sigma_t} [\pi_{\mathfrak{a}} \dot{\psi}^{\mathfrak{a}} - L(j^1\psi, \dot{\psi})\xi^0]\, \mathrm{d}s_0 \\ &\equiv H_\Xi(\psi, \pi), \end{aligned}$$

as claimed. □

Finally, if we denote by the same symbol $\omega_\Sigma := \omega_{\Sigma_t}$ the pull-back onto $\mathbf{\Phi}_t$ of the canonical 2-form on $T^*(\mathcal{P}_\lambda)_t$, dynamics can be described by the classical equation

$$X \lrcorner \omega_\Sigma = \mathrm{d}H_\Xi, \tag{5.3.6}$$

which is to be solved for (the flow of) the evolution vector field X. The local expression of (5.3.6) is known, of course, as ***Hamilton's equations***. Explicitly,

$$\begin{aligned} \dot{\psi}^{\mathfrak{a}} &= X^{\mathfrak{a}} = \tfrac{\delta H_\Xi}{\delta \pi_{\mathfrak{a}}}, \\ \dot{\pi}_{\mathfrak{a}} &= X_{\mathfrak{a}} = -\tfrac{\delta H_\Xi}{\delta \psi^{\mathfrak{a}}}, \end{aligned}$$

[2] The interested reader is referred to Gotay *et al.* (1999).

where locally $X = X^{\mathfrak{a}}\partial_{\mathfrak{a}} + X_{\mathfrak{a}}\partial^{\mathfrak{a}}$, $\partial_{\mathfrak{a}} := \partial/\partial\psi^{\mathfrak{a}}$, $\partial^{\mathfrak{a}} := \partial/\partial\pi_{\mathfrak{a}}$.

5.4 Bi-instantaneous dynamics

Dynamics with two evolution directions, or *bi-instantaneous dynamics*, is essentially analogous to dynamics with one evolution direction, but, for each configuration field, there are two velocity fields, and consequently two momentum fields. The initial surfaces are two (usually null) intersecting hypersurfaces of codimension 1, $\mathscr{N}_-$ and $\mathscr{N}_+$, and their (compact, orientable) intersection $\mathscr{S}$. Alternatively, one could consider any Cauchy hypersurface Σ of codimension 1 consisting of a one-parameter family of hypersurfaces of codimension 2 stretching between $\mathscr{N}_-$ and $\mathscr{N}_+$. The latter is the framework of Epp (1995), and is essentially analogous to the instantaneous formalism described in the previous section. Here, we are interested in developing a *characteristic* description of bi-instantaneous Hamiltonian dynamics along the lines of Hayward (1993).

The idea is to start with a (degenerate) multisymplectic structure on the i-th hypersurface $\mathscr{N}_i$, $i = -, +$, and then mimic the procedure we followed in the previous section for the transition from the multisymplectic to the instantaneous Hamiltonian description. As before, in the sequel we shall always assume to have chosen coordinates $(x^\mu)_{\mu=0}^{m-1} = (x^i, x^\alpha)_{\alpha=2}^{m-1}$ on M adapted to $\mathscr{N}_-$, $\mathscr{N}_+$ and $\mathscr{S}$, in the sense that $\mathscr{N}_-$, $\mathscr{N}_+$ and $\mathscr{S}$ are locally given by $x^+ = 0$, $x^- = 0$ and $x^+ = 0 = x^-$, respectively.

Now, let P^i_λ be the restriction of a gauge-natural bundle P_λ over M to the i-th hypersurface $\mathscr{N}_i$, and let $P^i_\lambda|_{\mathscr{S}}$ denote its restriction to $\mathscr{S}$. As usual, $\mathcal{P}^i_\lambda|_{\mathscr{S}}$ will denote the space of all the sections of $P^i_\lambda|_{\mathscr{S}}$. Accordingly, Z^i_λ shall denote the (homogeneous) Legendre bundle over P^i_λ (and $\mathscr{N}_i$), $Z^i_\lambda|_{\mathscr{S}}$ its restriction to $\mathscr{S}$, and $\mathcal{Z}^i_\lambda|_{\mathscr{S}}$ the space of its sections. Then, in analogy with the instantaneous case, we can define the canonical 1- and 2-forms on $\mathcal{Z}^i_\lambda|_{\mathscr{S}}$ by

$$u \lrcorner \Theta^i_{\mathscr{S}} = \int_\Sigma \sigma^*(u \lrcorner \Theta^i) \tag{5.4.1a}$$

and

$$v \lrcorner u \lrcorner \Omega^i_{\mathscr{S}} = \int_\Sigma \sigma^*(v \lrcorner u \lrcorner \Omega^i) \equiv -\mathrm{d}\Theta^i_{\mathscr{S}}, \tag{5.4.1b}$$

respectively, Θ^i and Ω^i being the canonical forms on Z^i_λ defined as in §5.2. As before, we define a vector bundle morphism $R^i_{\mathscr{S}} \colon \mathcal{Z}^i_\lambda|_{\mathscr{S}} \to (T^*\mathcal{P}_\lambda|_{\mathscr{S}})^2 := T^*\mathcal{P}^-_\lambda|_{\mathscr{S}} \oplus_{\mathcal{P}^-_\lambda|_{\mathscr{S}} \cong \mathcal{P}^+_\lambda|_{\mathscr{S}}} T^*\mathcal{P}^+_\lambda|_{\mathscr{S}}$ by

$$\langle R^i_{\mathscr{S}}(\sigma), \iota_i(v)\rangle := \int_\Sigma \Big(\pi_i \circ \sigma\Big)^*(v \lrcorner \sigma), \tag{5.4.2}$$

$\pi_i \colon Z^i_\lambda \to P^i_\lambda$ being the canonical projection, v an element of $T_{(\pi_i \circ \sigma)}\mathcal{P}^i_\lambda|_{\mathscr{S}}$, and $\iota_i \colon T\mathcal{P}^i_\lambda|_{\mathscr{S}} \to (T\mathcal{P}_\lambda|_{\mathscr{S}})^2 := T\mathcal{P}^-_\lambda|_{\mathscr{S}} \oplus_{\mathcal{P}^-_\lambda|_{\mathscr{S}} \cong \mathcal{P}^+_\lambda|_{\mathscr{S}}} T\mathcal{P}^+_\lambda|_{\mathscr{S}}$ the canonical vector bundle embedding. In adapted coordinates, $R^i_{\mathscr{S}}(\sigma)$ reads

$$R^i_{\mathscr{S}}(\sigma) = p_{\mathfrak{a}}{}^i \, \mathrm{d}y^{\mathfrak{a}} \otimes \mathrm{d}S,$$

where $\mathrm{d}S := \partial_i \lrcorner \mathrm{d}s$ and $\mathrm{d}s := \mathrm{d}x^i \wedge \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^{m-1}$. As before, we have

$$\Theta^i_{\mathscr{S}} = (R^i_{\mathscr{S}})^* \theta^i_{\mathscr{S}}, \tag{5.4.3a}$$

$$\Omega^i_{\mathscr{S}} = (R^i_{\mathscr{S}})^* \omega^i_{\mathscr{S}}, \tag{5.4.3b}$$

where $\theta^i_{\mathscr{S}}$ and $\omega^i_{\mathscr{S}}$ are the canonical 1- and 2-forms on $(T^*\mathcal{P}_\lambda|_{\mathscr{S}})^2$, respectively, defined by

$$\langle \theta^i_{\mathscr{S}}(\psi, \pi^-, \pi^+), v\rangle = \textstyle\int_{\mathscr{S}} \pi^i (T\pi_{\mathcal{P}^i_\lambda} \circ v), \tag{5.4.4a}$$

$$\omega^i_{\mathscr{S}} := -\mathrm{d}\theta^i_{\mathscr{S}}, \tag{5.4.4b}$$

(ψ, π^-, π^+) denoting a point in $(T^*\mathcal{P}_\lambda|_{\mathscr{S}})^2$, v an element of $T_{(\psi,\pi^-,\pi^+)}(T^*\mathcal{P}_\lambda|_{\mathscr{S}})^2$, and $\pi_{\mathcal{P}^i_\lambda} : (T^*\mathcal{P}_\lambda|_{\mathscr{S}})^2 \to \mathcal{P}^i_\lambda|_{\mathscr{S}}$ being the canonical projection. Locally,

$$\theta^i_{\mathscr{S}}(\psi, \pi^-, \pi^+) = \textstyle\int_{\mathscr{S}} \pi^i{}_{\mathfrak{a}}\, \mathrm{d}\psi^{\mathfrak{a}} \otimes \mathrm{d}S,$$
$$\omega^i_{\mathscr{S}}(\psi, \pi^-, \pi^+) = \textstyle\int_{\mathscr{S}} \mathrm{d}\psi^{\mathfrak{a}} \wedge \mathrm{d}\pi^i{}_{\mathfrak{a}} \otimes \mathrm{d}S.$$

Indeed, let $v \in T_\sigma \mathcal{Z}^i_\lambda|_{\mathscr{S}}$. By definition of pull-back (*cf.* §1.1) and (5.4.4a),

$$\begin{aligned}\langle (R^i_{\mathscr{S}})^* \theta^i_{\mathscr{S}}(\sigma), v\rangle &= \langle \theta^i_{\mathscr{S}}(R^i_{\mathscr{S}}(\sigma)), TR^i_{\mathscr{S}} \circ v\rangle \\ &= \langle R^i_{\mathscr{S}}(\sigma), T\pi_{\mathcal{P}^i_\lambda} \circ TR^i_{\mathscr{S}} \circ v\rangle.\end{aligned}$$

However, since $R^i_{\mathscr{S}}$ covers the identity,

$$\pi_{\mathcal{P}^i_\lambda} \circ R^i_{\mathscr{S}} = \tilde{\pi}_i,$$

where $\tilde{\pi}_i \colon \mathcal{Z}^i_\lambda|_{\mathscr{S}} \to \mathcal{P}^i_\lambda|_{\mathscr{S}}$ is the canonical projection. Hence,

$$T\pi_{\mathcal{P}^i_\lambda} \circ TR^i_{\mathscr{S}} \circ v = T\tilde{\pi}_i \circ v = T\pi_i \circ v.$$

Therefore,

$$\begin{aligned}\langle (R^i_{\mathscr{S}})^* \theta^i_{\mathscr{S}}(\sigma), v\rangle &= \langle R^i_{\mathscr{S}}(\sigma), T\pi_i \circ v\rangle \\ &= \textstyle\int_{\mathscr{S}} \left(\pi_i \circ \sigma\right)^* ((T\pi_i \circ v) \lrcorner \sigma) \\ &= \textstyle\int_{\mathscr{S}} \sigma^* (v \lrcorner \pi_i^* \sigma),\end{aligned}$$

where we used (5.4.2). However, by definition of Θ^i, $\pi_i^*\sigma = \Theta^i \circ \sigma$ [*cf.* (5.2.8) and (5.2.10)]. Thus, by (5.4.1a)

$$\langle (R^i_{\mathscr{S}})^* \theta^i_{\mathscr{S}}(\sigma), v\rangle = \langle \Theta^i_{\mathscr{S}}(\sigma), v\rangle,$$

which proves (5.4.3a), whereas (5.4.3b) follows from (5.4.1b), (5.4.3a) and (5.4.4b). □

Remark 5.4.1. As a mathematical curiosity, notice that, if we let $(\omega^i_{\mathscr{S}})^\#$ denote the endomorphism associated with $\omega^i_{\mathscr{S}}$, $i = -, +$, then we would find

$$(\omega^i_{\mathscr{S}})^\#_y = \int_{\mathscr{S}} f^i \delta^{m-1}(x-y) \otimes \mathrm{d}S_x,$$

where

$$f^- := \begin{pmatrix} 0 & \mathsf{I}_n & 0 \\ -\mathsf{I}_n & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad f^+ := \begin{pmatrix} 0 & 0 & \mathsf{I}_n \\ 0 & 0 & 0 \\ -\mathsf{I}_n & 0 & 0 \end{pmatrix},$$

$x, y \in \mathscr{S}$ and $n := \dim P_\lambda - \dim M$. Each $(\omega^i_{\mathscr{S}})^\#$ defines an *infinite-dimensional f-structure* (Yano 1963; Yano & Kon 1984). Moreover,

$$(\omega^-_{\mathscr{S}})^\# (\omega^+_{\mathscr{S}})^\# = [(\omega^+_{\mathscr{S}})^\# (\omega^-_{\mathscr{S}})^\#]^\top.$$

Now, to discuss dynamics, we need to modify the concept of a "slicing" introduced earlier on since we have two "times" here. Explicitly, we shall require that there exists a diffeomorphism $\tilde{s}_M \colon \mathscr{S} \times \mathbb{R}^2 \to M$, where $\mathscr{S}$ is a compact, orientable $(m-2)$-dimensional and M is our m-dimensional space-time. Following the literature on the subject, we shall call the pair $(\mathscr{S}, \tilde{s}_M)$ a ***double-null slicing***.

For $(x^-, x^+) \in \mathbb{R}^2$, we shall write $\mathscr{S}_\mp$ for $s_M(\mathscr{S} \times \{(x^-, x^+)\})$. Also, we shall usually denote by ξ and η the generators of $\tilde{s}_M$, i.e. $\xi := (\tilde{s}_M)_*(\partial/\partial x^-)$ and $\eta := (\tilde{s}_M)_*(\partial/\partial x^+)$. The concept of compatible double-null slicings of (gauge-natural) bundles over M can then be defined along the lines of the previous section, as well as their infinitesimal counterparts.

Fix now an infinitesimal double-null slicing $(\Xi_\lambda, \mathrm{H}_\lambda)$ of the configuration bundle P_λ of a first order gauge-natural field theory described by a Lagrangian $\mathcal{L} \colon J^1 P_\lambda \to \bigwedge^m T^* M$. The corresponding "slice" $(P_\lambda)_\mp$ of P_λ is, of course, a gauge-natural bundle over the "slice" $\mathscr{S}_\mp$ of M corresponding to the projections ξ and η of Ξ_λ and H_λ, respectively. In analogy with the instantaneous case, we can define two operators $\tilde{\mathscr{L}}_\Xi y_\mp$ and $\tilde{\mathscr{L}}_\mathrm{H} y_\mp$ by

$$\tilde{\mathscr{L}}_\Xi y_\mp \circ j^1_x \sigma \equiv \tilde{\mathscr{L}}_\Xi \sigma(x) \qquad \text{and} \qquad \tilde{\mathscr{L}}_\Xi y_\mp \circ j^1_x \sigma \equiv \tilde{\mathscr{L}}_\mathrm{H} \sigma(x)$$

for all $\sigma \in \mathcal{P}_\lambda$ and $x \in \mathscr{S}_\mp$, and we shall set

$$\psi := \sigma|_{\mathscr{S}_\mp}, \qquad \psi_- := \tilde{\mathscr{L}}_\Xi y_\mp \circ j^1\sigma, \qquad \psi_+ := \tilde{\mathscr{L}}_\mathrm{H} y_\mp \circ j^1\sigma.$$

The construction of the jet decomposition map proceeds as before. Explicitly, we define it to be the map $\beta_{\Xi,\mathrm{H}} \colon (J^1 P_\lambda)_\mp \to J^1(P_\lambda)_\mp \times (VP_\lambda)_\mp \times (VP_\lambda)_\mp$ such that

$$\beta_{\Xi,\mathrm{H}}(j^1_x\sigma) = (j^1_x\psi, \psi_-(x), \psi_+(x))$$

for all $\sigma \in \mathcal{P}_\lambda$ and $x \in \mathscr{S}_\mp$. In coordinates adapted to $\mathscr{S}_\mp$, this reads

$$\beta_{\Xi,\mathrm{H}}\Big((x^\mu, y^\mathfrak{a}, y^\mathfrak{a}{}_\nu\Big) \circ j^1_x\sigma) = \Big((x^\mu, y^\mathfrak{a}, y^\mathfrak{a}{}_\alpha) \circ j^1_x\psi, y^\mathfrak{a}{}_- \circ \psi_-(x), y^\mathfrak{a}{}_+ \circ \psi_+(x)\Big).$$

Hence, we can define the ***bi-instantaneous Lagrangian*** $L_{\Xi,\mathrm{H}} \colon (T(\mathcal{P}_\lambda)_\mp)^2 \to \mathbb{R}$ by

$$L_{\Xi,\mathrm{H}}(\psi, \psi_-, \psi_+) \equiv \int_{\mathscr{S}_\mp} \mathcal{L}_{\Xi,\mathrm{H}}(\psi, \psi_-, \psi_+) = \int_{\mathscr{S}_\mp} i^*_{\mathscr{S}}\Big(\eta \,\lrcorner\, (\xi \,\lrcorner\, \mathcal{L}(j^1\sigma))\Big),$$

for $(\psi, \psi_-, \psi_+) \in (T(\mathcal{P}_\lambda)_\mp)^2$, where $j^1\sigma \circ i_{\mathscr{S}}$ is the reconstruction of $(j^1\psi, \psi_-, \psi_+)$ and

$i_{\mathscr{S}}\colon \mathscr{S}_{\mp} \to M$ is the inclusion. In coordinates adapted to $\mathscr{S}_{\mp}$, it reads

$$L_{\Xi,\mathrm{H}}(\psi, \psi_-, \psi_+) = 2\int_{\mathscr{S}_{\mp}} L(j^1\psi, \psi_-, \psi_+)\xi^{[-}\eta^{+]}\,\mathrm{d}S,$$

where (consistently) $\mathrm{d}S = \mathrm{d}s_{-+} \equiv \mathrm{d}x^2 \wedge \cdots \wedge \mathrm{d}x^{m-1}$. The bi-instantaneous Lagrangian $L_{\Xi,\mathrm{H}}$ has a ***bi-instantaneous Legendre transform***

$$\begin{cases} \mathbb{F}L_{\Xi,\mathrm{H}}\colon (T(\mathcal{P}_\lambda)_{\mp})^2 \to (T^*(\mathcal{P}_\lambda)_{\mp})^2 \\ \mathbb{F}L_{\Xi,\mathrm{H}}\colon (\psi, \psi_-, \psi_+) \mapsto (\psi, \pi^-, \pi^+) \end{cases},$$

defined in the usual manner. In adapted coordinates,

$$\pi^i = \pi_{\mathfrak{a}}{}^i\,\mathrm{d}y^{\mathfrak{a}} \otimes \mathrm{d}S,$$

where

$$\pi_{\mathfrak{a}}{}^i \circ (\psi, \psi_-, \psi_+) \equiv \frac{\partial \mathcal{L}_{\Xi,\mathrm{H}}}{\partial y^{\mathfrak{a}}_i} \circ (\psi, \psi_-, \psi_+).$$

In analogy with instantaneous case, we shall call

$$\mathbf{\Phi}_{\mp} := \mathbb{F}L_{\Xi,\mathrm{H}}\Big((T(\mathcal{P}_\lambda)_{\mp})^2\Big) \subseteq (T^*(\mathcal{P}_\lambda)_{\mp})^2$$

the ***bi-instantaneous primary constraint set***. On $\mathbf{\Phi}_{\mp}$, we can then define the ***bi-instantaneous Hamiltonian*** $H_{\Xi,\mathrm{H}}$ by

$$H_{\Xi,\mathrm{H}}(\psi, \pi^-, \pi^+) = \langle \pi^i, \psi_i \rangle - L_{\Xi,\mathrm{H}}(\psi, \psi_-, \psi_+).$$

In coordinates,

$$H_{\Xi,\mathrm{H}}(\psi, \pi^-, \pi^+) \equiv \int_{\mathscr{S}_{\mp}} \mathcal{H}_{\Xi,\mathrm{H}}(\psi, \pi^-, \pi^+) = \int_{\mathscr{S}_{\mp}} \Big(\pi_{\mathfrak{a}}{}^i\psi^{\mathfrak{a}}{}_i - 2L(j^1\psi, \psi_-, \psi_+)\xi^{[-}\eta^{+]}\Big)\,\mathrm{d}S.$$

Finally, if we denote by the same symbol $\omega^i_{\mathscr{S}} := \omega^i_{\mathscr{S}_{\mp}}$ the pull-back onto $\mathbf{\Phi}_{\mp}$ of the i-th canonical 2-form on $(T^*(\mathcal{P}_\lambda)_{\mp})^2$, dynamics can be described by the equation

$$X_i \lrcorner\, \omega^i_{\mathscr{S}} = \mathrm{d}H_{\Xi,\mathrm{H}}, \tag{5.4.5}$$

which is to be solved for the i-th evolution vector field X_i. Locally, (5.4.5) reads

$$\begin{aligned} X_i \lrcorner\, \omega^i_{\mathscr{S}} &= [X_i{}^{\mathfrak{a}}\partial_{\mathfrak{a}} + X_{i\mathfrak{a}}{}^j\partial^{\mathfrak{a}}{}_j] \lrcorner \textstyle\int_{\mathscr{S}_{\mp}} \mathrm{d}\psi^{\mathfrak{b}} \wedge \mathrm{d}\pi_{\mathfrak{b}}{}^i \otimes \mathrm{d}S \\ &= \textstyle\int_{\mathscr{S}_{\mp}} (X_i{}^{\mathfrak{a}}\,\mathrm{d}\pi_{\mathfrak{a}}{}^i - X_{i\mathfrak{a}}{}^i\,\mathrm{d}\psi^{\mathfrak{a}}) \otimes \mathrm{d}S \\ &= \mathrm{d}H_{\Xi,\mathrm{H}} = \textstyle\int_{\mathscr{S}_{\mp}} \Big(\frac{\delta H_{\Xi,\mathrm{H}}}{\delta \psi^{\mathfrak{a}}}\,\mathrm{d}\psi^{\mathfrak{a}} + \frac{\delta H_{\Xi,\mathrm{H}}}{\delta \pi_{\mathfrak{a}}{}^i}\,\mathrm{d}\pi_{\mathfrak{a}}{}^i\Big) \otimes \mathrm{d}S, \end{aligned}$$

which implies

$$\begin{aligned} \psi^{\mathfrak{a}}{}_i &= X_i{}^{\mathfrak{a}} = \tfrac{\delta H_{\Xi,\mathrm{H}}}{\delta \pi_{\mathfrak{a}}{}^i}, \\ \pi_{\mathfrak{a}}{}^i{}_i &= X_{i\mathfrak{a}}{}^i = -\tfrac{\delta H_{\Xi,\mathrm{H}}}{\delta \psi^{\mathfrak{a}}}, \end{aligned}$$

which are precisely the ***bi-instantaneous Hamilton equations*** (Hayward 1993). From these we see that the trace-free part of $(X_{i\mathfrak{a}}{}^{j})$ is not determined. This is related to a qualitatively new feature of bi-instantaneous dynamics, namely that the above Hamilton equations do not by themselves give the full (first-order) field equations, but need to be supplemented by an additional condition, such as the *integrability condition*

$$[\Xi, \mathrm{H}] = 0. \tag{5.4.6}$$

(*cf.*, e.g., Hayward 1993). Of course, equation (5.4.6) implies $[\xi, \eta] \equiv T\pi \circ [\Xi, \mathrm{H}] = 0$, which in turn implies that, locally, $\xi = \delta^{\mu}{}_{-}\partial_{\mu}$ and $\eta = \delta^{\mu}{}_{+}\partial_{\mu}$, $\pi\colon P \to M$ denoting here the canonical projection. By virtue of Proposition 1.10.14 condition (5.4.6) also entails $[\Xi_{\lambda}, \mathrm{H}_{\lambda}] = 0$, which in turn implies $[\pounds_{\Xi}, \pounds_{\mathrm{H}}] = 0$ on any (gauge-natural) *vector* or *affine* bundle $P_{\lambda} \to M$ because of (2.2.5). This result is often very useful in practice.

To illustrate how this works in actual fact, we shall give the following simple

Example 5.4.2 (Klein-Gordon scalar field in Minkowski space-time). The dynamics of an $\mathbb{R}$-valued Klein-Gordon scalar field ψ in Minkowski space-time $(\mathbb{R}^4, \eta)$ is described by the ***Klein-Gordon Lagrangian***

$$\begin{cases} \mathfrak{L}_{\mathrm{KG}}\colon J^1(\mathbb{R}^4 \times \mathbb{R}) \to \bigwedge^4 T^*\mathbb{R}^4 \\ \mathfrak{L}_{\mathrm{KG}}\colon j^1\psi \mapsto \mathfrak{L}_{\mathrm{KG}}(j^1\psi) := \frac{1}{2}(\partial_{\mu}\psi\partial^{\mu}\psi - m^2\psi^2) \end{cases},$$

indices being lowered and raised by η and η^{-1}, respectively. The corresponding Euler-Lagrange equations are readily found to be

$$e(\mathfrak{L}_{\mathrm{KG}}) \circ j^2\psi \equiv -(\partial_{\mu}\partial^{\mu} + m^2)\psi = 0. \tag{5.4.7}$$

Now, any infinitesimal double-null slicing $(\Xi_{\lambda}, \mathrm{H}_{\lambda})$ of the natural vector bundle $\mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ is simply given by an infinitesimal slicing (ξ, η) of the basis $\mathbb{R}^4$. The bi-instantaneous Klein-Gordon Lagrangian then reads

$$L_{\Xi,\mathrm{H}}(\psi, \psi_{-}, \psi_{+}) = 2\int_{\mathscr{S}_{\mp}} \left(\psi_{-}\psi_{+} + \frac{1}{2}\partial_{\alpha}\psi\partial^{\alpha}\psi - \frac{1}{2}m^2\psi^2\right)\xi^{[-}\eta^{+]}\,\mathrm{d}S.$$

Thus, the bi-instantaneous Legendre transform is

$$\pi^{-} = \frac{\partial\mathfrak{L}_{\Xi,\mathrm{H}}}{\partial\psi_{-}} = 2\xi^{[-}\eta^{+]}\psi_{+}, \qquad \pi^{+} = \frac{\partial\mathfrak{L}_{\Xi,\mathrm{H}}}{\partial\psi_{+}} = 2\xi^{[-}\eta^{+]}\psi_{-},$$

whence the bi-instantaneous Hamiltonian reads

$$H_{\Xi,\mathrm{H}}(\psi, \pi_{-}, \pi_{+}) = \int_{\mathscr{S}_{\mp}} \left[\frac{1}{\xi^{[-}\eta^{+]}}\pi^{-}\pi^{+} - \left(\frac{1}{2}\partial_{\alpha}\psi\partial^{\alpha}\psi - \frac{1}{2}m^2\psi^2\right)\xi^{[-}\eta^{+]}\right]\mathrm{d}S$$

and the Hamilton equations are

$$\psi_{-} \approx \tfrac{\delta H_{\Xi,\mathrm{H}}}{\delta\pi^{-}} = \tfrac{1}{\xi^{[-}\eta^{+]}}\pi^{+}, \qquad \psi_{+} \approx \tfrac{\delta H_{\Xi,\mathrm{H}}}{\delta\pi^{+}} = \tfrac{1}{\xi^{[-}\eta^{+]}}\pi^{-},$$
$$\pi^{-}{}_{-} + \pi^{+}{}_{+} \approx -\tfrac{\delta H_{\Xi,\mathrm{H}}}{\delta\psi} = -2\xi^{[-}\eta^{+]}(\partial_{\alpha}\partial^{\alpha} + m^2)\psi.$$

Now, here integrability condition (5.4.6) simply reduces to $[\xi, \eta] = 0$. As we saw, this entails, on one hand, $\xi = \delta^\mu{}_-\partial_\mu$ and $\eta = \delta^\mu{}_+\partial_\mu$, which acquires a *canonical* meaning here since we are in $\mathbb{R}^4$. On the other hand, on using $\xi^\mu = \delta^\mu{}_-$ and $\eta^\mu = \delta^\mu{}_+$, it yields

$$0 = \pounds_{[\xi,\eta]}\psi \equiv [\pounds_\xi, \pounds_\eta]\psi = \psi_{+-} - \psi_{-+} \approx \pi^-{}_- - \pi^+{}_+.$$

Thus, the Hamilton equations together with the integrability condition give

$$\psi_- \approx \pi^+, \qquad \psi_+ \approx \pi^-, \qquad \pi^-{}_- \approx \pi^+{}_+ \approx -\frac{1}{2}(\partial_\alpha\partial^\alpha + m^2)\psi,$$

and the initial data are ψ on $\mathscr{S}_\mp$, π^- on $\mathscr{N}_-$ and π^+ on $\mathscr{N}_+$. On recalling, now, the explicit expression for ξ and η, it is easy to realize that these equations are indeed equivalent to Euler-Lagrangian equations (5.4.7).

5.5 Future work

As we saw, integrability condition (5.4.6) is extremely useful in simplifying calculations, but, geometrically speaking, represents a very strong requirement. Although we believe that some sort of "compatibility condition" between the vector fields Ξ_λ and H_λ is necessary, we are currently investigating how to relax the integrability condition in actual fact (Matteucci & Vickers 2003).

Also, it is natural to ask whether a suitable definition of Poisson brackets can be given in the context of bi-instantaneous dynamics. An obvious candidate for the "i-th Poisson bracket" would be

$$\{f, g\}_i := (X_g)_i \lrcorner (X_f)_j \lrcorner \omega^j_{\mathscr{S}},$$

where the 'i-th Hamiltonian vector field' $(X_f)_i$ with respect to the function(al) f in $C^\infty((T^*\mathcal{P}_\lambda|_{\mathscr{S}})^2; \mathbb{R})$ is defined to be the vector field on $(T^*\mathcal{P}_\lambda|_{\mathscr{S}})^2$ such that

$$(X_f)_i \lrcorner \omega^i_{\mathscr{S}} = \mathrm{d}f$$

[*cf.* (5.4.5)]. Unfortunately, these brackets, though geometrically well-defined, are algebraically awkward, i.e. they do not possess any clear-cut symmetry. In particular, they are not antisymmetric and do not form a Lie algebra.

In this connection, we believe that only a correct understanding—still lacking at present—of the concept of Poisson brackets in multisymplectic geometry could provide full insight into this matter.

Conclusions and perspectives

> Die ganze Zahl schuf der liebe Gott, alles Übrige ist Menschenwerk.
>
> L. Kronecker, in: F. Cajori, *A history of mathematics*

In a remarkable paper published in 1972, Andrzej Trautman introduced the notion of a generalized Lie derivative, and, arguably for the first time, formulated the theory of conserved quantities for first order Lagrangians in the elegant and concise language of modern differential geometry. He wrote:

> A general lesson to be drawn from the development of the theory of relativity is that it is desirable to analyse in detail the various structures inherent in the mathematical models used to describe physical phenomena. [...] With respect to differential-geometric notions, the custom of expressing everything in coordinates prevailed for a long time. [...] a steady and stubborn use of coordinates makes it difficult to separate and describe the various geometrical structures associated with physical theories. An important example of such a situation is provided by the variational principles of physics.

In a later and equally remarkable paper, Trautman (1980) reviewed the important role fibre bundles play in present-day physics, especially in connection with the Einstein-Cartan theory. In particular, he observed:

> The most important difference between gravitation and other gauge theories is due to the soldering of the bundle of frames LM to the base manifold M. The bundle LM is constructed in a natural and unique way from M, whereas a noncontractible M may be the base of inequivalent bundles with the same structure group. [...] What is the structure group G of the gravitational principal bundle P? Since space-time M is four dimensional, if $P = LM$ then $G = \mathrm{GL}(4, \mathbb{R})$. But one could equally well take for P the bundle AM of affine frames; in this case G is the affine group. [...] If one assumes—as usually one does—that ω and g are compatible, then the structure group of LM or AM can be restricted to the Lorentz or the Poincaré group, respectively.

In many respects, this thesis is the ideal continuation of that work, and sheds some new light on the issues raised by Trautman by using Trautman's very formalism. In particular, his definition of a generalized Lie derivative turned out to be the key to understand and systematize the hoary problem of Lie differentiation of spinor fields, and hence suggest that the theory of conserved quantities associated with the gravitational field coupled with Dirac fields could be properly analysed only in a truly gauge-natural context.

In §4.6 we observed that the Kosmann lift, which arises almost "naturally" in the theory of the Lie derivatives of spinors, seems to play a privileged role in the theory of conserved quantities too. It would be interesting to investigate to what extent this is true, and whether other lifts could be more suitable in the description of different physical situations.

In Chapter 5 we saw that gauge-naturality and generalized Lie derivatives arise naturally in both multisymplectic geometry and the intrinsic formulation of instantaneous and bi-instantaneous dynamics. As we mentioned in §5.5, there are several interesting open problems in this area, and there seems to be much to be gained in the study of field theory from the rich geometric structure of the multisymplectic formalism.

Appendix A

Categories and functors

In the sequel, we shall regard the concept of a *class* as primitive. Intuitively, a class is a collection whose elements are sets and/or elements of sets.

A.1 Categories

Definition A.1.1. The elements of a class are called ***objects***.

Definition A.1.2. Let $\mathcal{X}$ and $\mathcal{X}'$ be two classes. By a ***map*** (***between classes***) we shall mean a rule assigning to each object in $\mathcal{X}$ an object in $\mathcal{X}'$.

Definition A.1.3. A class $\mathcal{X}$ is called a ***set*** if there exists at least another class $\mathcal{Y}$ such that $\mathcal{X} \in \mathcal{Y}$.

Definition A.1.4. Let X and Y be two sets. By a ***map*** (***between sets***) with *domain* X and *codomain* Y we shall mean a rule assigning to each element $x \in X$ an element $f(x) \in Y$. Then, we shall write $f\colon X \to Y$ or $f\colon x \mapsto f(x)$.

Definition A.1.5. A map (between sets) f is said to ***equal*** another map (between sets) g, and we write $f = g$, if f and g have the same domain, the same codomain and the same value $f(x) = g(x)$ for any element x of their common domain.

Definition A.1.6. Let X be a set. The map

$$\begin{cases} \mathrm{id}_X\colon X \to X \\ \mathrm{id}_X\colon x \mapsto \mathrm{id}_X(x) := x \end{cases}$$

is called the ***identity*** (***map***) on X.

Definition A.1.7. Let X, Y, Z three sets and $f\colon X \to Y$, $g\colon Y \to Z$ two maps. By the ***composite map*** of g and f we shall mean the map

$$\begin{cases} g \circ f\colon X \to Z \\ g \circ f\colon x \mapsto (g \circ f)(x) := g(f(x)) \end{cases}.$$

Definition A.1.8. A map $f\colon X \to Y$ is said to be ***injective*** (or an ***injection*** or "***into***") whenever $x_1 \neq x_2$ in X implies $f(x_1) \neq f(x_2)$ in Y.

Definition A.1.9. A map $f\colon X \to Y$ is said to be ***surjective*** (or a ***surjection*** or "***onto***") whenever for each $y \in Y$ there exists at least an $x \in X$ such that $f(x) = y$.

Definition A.1.10. A map $f\colon X \to Y$ is said to be ***bijective*** (or a ***bijection*** or "***one-to-one***") if it is both injective and surjective, i.e. whenever for each $y \in Y$ there exists only one $x \in X$ such that $f(x) = y$.

Definition A.1.11. Let X and Y be two sets and $f\colon X \to Y$ and $g\colon Y \to X$ two maps. If $g \circ f = \mathrm{id}_X$, g is said to be a ***left-sided inverse*** of f. If $f \circ g = \mathrm{id}_Y$, g is said to be a ***right-sided inverse*** of f. If g is both a left-sided and a right-sided inverse of f, it is called a ***two-sided inverse*** of f.

Theorem A.1.12. *A map with non-empty domain is an injection iff it has a left-sided inverse, and is a surjection iff it has a right-sided inverse.*

Corollary A.1.13. *Let X and Y be two sets and $f\colon X \to Y$ a map. Then, the following statements are equivalent:*

(i) *f is a bijection;*

(ii) *f has both a left-sided and a right-sided inverse;*

(iii) *f has a two-sided inverse.*

In this case, any two inverses (left-sided, right-sided or two-sided) coincide. Such a unique inverse of f (denoted by f^{-1}) is bijective and, furthermore,

$$(f^{-1})^{-1} = f.$$

Definition A.1.14. The unique inverse f^{-1} of f as defined in the previous corollary is called the ***inverse*** (***map***) of f, and f is said to be *invertible*.

Definition A.1.15. By the ***direct product*** $X_1 \times X_2 \times \cdots \times X_n$ of n sets $X_1, X_2, \ldots, X_n$ we shall mean the set of all *ordered n-tuples* of elements of $X_1, X_2, \ldots, X_n$, respectively, i.e. the set

$$X_1 \times X_2 \times \cdots \times X_n := \{\, (x_1, x_2, \ldots, x_n) \mid x_1 \in X_1,\ x_2 \in X_2, \ldots,\ x_n \in X_n \,\}.$$

Definition A.1.16. Let n be a natural number, X a set and

$$X^n := \begin{cases} \underbrace{X \times \cdots \times X}_{n \text{ times}} & \text{if } n \neq 0, \\ \{1\} & \text{if } n = 0 \end{cases}$$

the n-th direct product of X with itself. By an ***n-ary operation*** we shall mean a map $f\colon X^n \to X$.

Definition A.1.17. Let X and X' be two sets equipped with an n-ary operation $h\colon X^n \to X$ and an n-ary operation $h'\colon (X')^n \to X'$, respectively. By a ***morphism*** or, more precisely, a ***homomorphism***[1] (***of*** n***-ary operations***) we shall mean a map $f\colon X \to X'$ such that

$$(f \circ h)(x_1, x_2, \ldots, x_n) = h'(f(x_1), f(x_2), \ldots, f(x_n))$$

for all $x_1, x_2, \ldots, x_n \in X$.

We shall denote the set of all morphisms between X and X' by $\mathrm{Mor}(X, X')$. We shall say that a morphism $f\colon X \to X'$ is a *monomorphism* if the map f is an injection, an *epimorphism* if it is a surjection, an *isomorphism* if it is a bijection and f^{-1} is a morphism.

A morphism $f\colon X \to X$ is called an *endomorphism* (of X) or, if it is bijective, an *automorphism* (of X).

Definition A.1.18. By a ***category*** we shall mean a pair $\boldsymbol{X} := (\mathrm{Ob}(\boldsymbol{X}), \mathrm{Mor}(\boldsymbol{X}))$, where $\mathrm{Ob}(\boldsymbol{X})$ is a class and $\mathrm{Mor}(\boldsymbol{X})$ is the class formed by all the sets $\mathrm{Mor}(X, Y)$, $X, Y \in \mathrm{Ob}(\boldsymbol{X})$, satisfying the following two conditions:

(*i*) $\mathrm{id}_X \in \mathrm{Mor}(X, X)$ for all $X \in \mathrm{Ob}(\boldsymbol{X})$;

(*ii*) if $f \in \mathrm{Mor}(X, Y)$ and $g \in \mathrm{Mor}(Y, Z)$, then $g \circ f \in \mathrm{Mor}(X, Z)$ for all $X, Y, Z \in \mathrm{Ob}(\boldsymbol{X})$.

The objects of $\mathrm{Ob}(\boldsymbol{X})$ are simply called the *objects* of the category $\boldsymbol{X}$, whereas the objects of $\mathrm{Mor}(\boldsymbol{X})$ are said to be the *morphisms* of $\boldsymbol{X}$.

In this thesis, three main categories are considered: the category $\boldsymbol{Mf}_m$ of m-dimensional manifolds and local diffeomorphisms, the category $\boldsymbol{FM}$ of fibred manifolds and fibre-respecting morphisms and, finally, the category $\boldsymbol{PB}_m(G)$ of principal G-bundles over m-dimensional manifolds and principal bundle morphisms. It is easy to realize that these are indeed categories in the sense of Definition A.1.18.

A.2 Functors

Definition A.2.1. Let $\boldsymbol{X}$ and $\boldsymbol{X}'$ be two categories. By a (***covariant***) ***functor*** from $\boldsymbol{X}$ to $\boldsymbol{X}'$ we shall mean a map $\mathscr{F}\colon \boldsymbol{X} \to \boldsymbol{X}'$ assigning to each object X in $\boldsymbol{X}$ an object $\mathscr{F}(X)$ in $\boldsymbol{X}'$ and to each morphism $f\colon X \to Y$ in $\boldsymbol{X}$ a morphism $\mathscr{F}(f)\colon \mathscr{F}(X) \to \mathscr{F}(Y)$ in $\boldsymbol{X}'$, and satisfying the following two conditions:

(*i*) $\mathscr{F}(\mathrm{id}_X) = \mathrm{id}_{\mathscr{F}(X)}$ for all $X \in \mathrm{Ob}(\boldsymbol{X})$;

(*ii*) $\mathscr{F}(g \circ f) = \mathscr{F}(g) \circ \mathscr{F}(f)$ for all $f \in \mathrm{Mor}(X, Y)$, $g \in \mathrm{Mor}(Y, Z)$, $X, Y, Z \in \mathrm{Ob}(\boldsymbol{X})$.

In this thesis, two important kinds of functors are considered: natural and gauge-natural functors (*cf.* Chapter 1).

[1]The term "morphism", although less precise, is commonly used in the general case, whereas the term "homomorphism" is mainly used for linear maps between vector spaces.

Definition A.2.2. Let $\boldsymbol{X}$ and $\boldsymbol{X}'$ be two categories. By a ***contravariant functor*** from $\boldsymbol{X}$ to $\boldsymbol{X}'$ we shall mean a map $\mathscr{F} : \boldsymbol{X} \to \boldsymbol{X}'$ assigning to each object X in $\boldsymbol{X}$ an object $\mathscr{F}(X)$ in $\boldsymbol{X}'$ and to each morphism $f \colon X \to Y$ in $\boldsymbol{X}$ a morphism $\mathscr{F}(f) \colon \mathscr{F}(Y) \to \mathscr{F}(X)$ in $\boldsymbol{X}'$, and satisfying condition (i) of Definition A.2.1 and, instead of condition (ii), the following one:

(ii') $\mathscr{F}(g \circ f) = \mathscr{F}(f) \circ \mathscr{F}(g)$ for all $f \in \mathrm{Mor}(X,Y)$, $g \in \mathrm{Mor}(Y,Z)$, $X, Y, Z \in \mathrm{Ob}(\boldsymbol{X})$.

Definition A.2.3. Let $\mathscr{F}, \mathscr{G} \colon \boldsymbol{X} \to \boldsymbol{X}'$ two functors. By a ***natural transformation*** from $\mathscr{F}$ to $\mathscr{G}$ we shall mean a map $\tau \colon \mathscr{F} \to \mathscr{G}$ assigning to each object X in $\boldsymbol{X}$ a morphism $\tau_X \colon \mathscr{F}(X) \to \mathscr{G}(X)$ in $\boldsymbol{X}'$ such that, for any morphism $f \colon X \to Y$ in $\boldsymbol{X}$, $\tau_Y \circ \mathscr{F}(f) = \mathscr{G}(f) \circ \tau_X$. Equivalently, the following diagram is commutative.

$$
\begin{array}{ccc}
\mathscr{F}(X) & \xrightarrow{\tau_X} & \mathscr{G}(X) \\
{\scriptstyle \mathscr{F}(f)}\downarrow & & \downarrow{\scriptstyle \mathscr{G}(f)} \\
\mathscr{F}(Y) & \xrightarrow[\tau_Y]{} & \mathscr{G}(Y)
\end{array}
$$

An analogous definition exists for contravariant functors. A natural transformation $\tau \colon \mathscr{F} \to \mathscr{G}$ is also known as a *functor morphism*.

Definition A.2.4. Let τ, $\boldsymbol{X}$, $\boldsymbol{X}'$, $\mathscr{F}$ and $\mathscr{G}$ be as in the previous definition. If τ_X is invertible in $\boldsymbol{X}'$ for all $X \in \mathrm{Ob}(\boldsymbol{X})$, we call τ a ***natural isomorphism*** (or a ***natural equivalence***), and we note that the inverses $\{(\tau_X)^{-1}\}$ form a natural isomorphism $\tau^{-1} \colon \mathscr{G} \to \mathscr{F}$. In this case, we write $\mathscr{F} \cong_\tau \mathscr{G}$ or, when clear from the context, $\mathscr{F} \cong \mathscr{G}$.

Appendix B

Vector fields and flows

In this appendix we collect together a number of classical results on vector fields and flows, which will prove useful in the rest of the thesis.

B.1 General definitions

Definition B.1.1. A ***vector field*** ξ on a manifold M is a smooth section of the tangent bundle, so $\xi\colon M \to TM$ is smooth and $\tau_M \circ \xi = \mathrm{id}_M$, $\tau_M\colon TM \to M$ denoting the (canonical) tangent bundle projection.

We shall denote the set of all vector fields on M by $\mathfrak{X}(M)$. With pointwise addition and scalar multiplication $\mathfrak{X}(M)$ becomes an infinite-dimensional vector space.

Proposition B.1.2. *$\mathfrak{X}(M)$ coincides canonically with the space of all derivations of the algebra $C^\infty(M;\mathbb{R})$ of smooth functions from M to $\mathbb{R}$, i.e. with the space of all $\mathbb{R}$-linear operators $D\colon C^\infty(M;\mathbb{R}) \to C^\infty(M;\mathbb{R})$ such that $D(fg) = D(f)g + fD(g)$ for all $f, g \in C^\infty(M;\mathbb{R})$.*

Proof. See, e.g., Kolář *et al.* (1993), p. 16. □

Definition B.1.3. The $\mathbb{R}$-bilinear mapping $[\cdot,\cdot]\colon \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined as

$$[\xi, \eta](f) := \xi(\eta(f)) - \eta(\xi(f))$$

for all $\xi, \eta \in \mathfrak{X}(M)$ and $f \in C^\infty(M;\mathbb{R})$ is called the ***commutator*** (or ***Lie bracket***) of ξ and η, and turns $\mathfrak{X}(M)$ into a Lie algebra (*cf.* Definition C.2.1).

Definition B.1.4. Let $I \subseteq \mathbb{R}$ be an interval and $\gamma\colon I \to M$ a smooth curve on a manifold M defined on I, and set $\dot\gamma(t) \equiv \frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) := T_t\gamma(1)$. Then, we shall say that γ is an ***integral curve*** of a vector field $\xi \in \mathfrak{X}(M)$ if $\dot\gamma(t) = \xi(\gamma(t))$ for all $t \in I$. In this case, we shall say that γ is *maximal* if either $I = \mathbb{R}$ or γ leaves M in finite (*parameter*) *time* t in the past, in the future, or both.

Theorem B.1.5 (Picard-Lindelöf). *Let ξ be a vector field on M. Then, for any x in M there is an open interval I_x containing 0 and an integral curve $\gamma_x\colon I_x \to M$ for ξ* based at x, *i.e. such that $\gamma_x(0) = x$. If γ_x is maximal, then it is unique.*

Definition B.1.6. Let ξ be a vector field on M and write $\varphi_t^\xi(x) \equiv \varphi^\xi(t,x) := \gamma_x(t)$, where $\gamma_x\colon I_x \to M$ is the maximal integral curve of ξ constructed in the previous theorem. The mapping φ^ξ or, equivalently, the set of maps $\{\varphi_t^\xi\}_{t\in I_x}$ will be called the ***flow*** of the vector field ξ.

In the rest of this thesis, we shall tend to omit the superscript ξ whenever it is clear which is the vector field whose flow we shall be considering.

Theorem B.1.7. *For each vector field ξ on M the mapping $\varphi^\xi\colon D_\xi \to M$, where $D_\xi := \bigcup_{x\in M}(I_x \times \{x\})$ is an open neighbourhood of $0 \times M$ in $\mathbb{R} \times M$, is smooth. Then, we have*

$$\varphi^\xi(t+s,x) = \varphi^\xi(t,\varphi^\xi(s,x))$$

in the following sense: if the r.h.s. exists, then the l.h.s. exists and we have equality; if t and s are both non-negative or both non-positive, and if the l.h.s. exists, then also the r.h.s. exists and we have equality.

Definition B.1.8. Let ξ be a vector field on M. Its flow φ^ξ is called *global* or *complete* if D_ξ defined in the previous theorem equals $\mathbb{R} \times M$. Then, φ_t^ξ is a diffeomorphism of M for all $t \in \mathbb{R}$ and ξ is called a ***complete vector field***.

Now, recall that the *support* supp_ξ of a vector field ξ on M is the closure of the set $\{x \in M \mid \xi(x) \neq 0\}$.

Proposition B.1.9. *Every vector field with compact support on M is complete.*

Corollary B.1.10. *On a compact manifold every vector field is complete.*

From Theorem B.1.7 and Definition B.1.6 it follows immediately that

$$\varphi_t^\xi \circ \varphi_s^\xi = \varphi_{t+s}^\xi, \tag{B.1.1a}$$

$$\varphi_0^\xi = \mathrm{id}_M, \tag{B.1.1b}$$

whence also

$$(\varphi_t^\xi)^{-1} = \varphi_{-t}^\xi, \tag{B.1.1c}$$

i.e. $\{\varphi_t^\xi\}$ is a (local) one-parameter group of diffeomorphisms of M. From Definitions B.1.6 and B.1.4 it also follows that

$$\frac{\partial}{\partial t}\varphi_t^\xi(x) = \xi \circ \varphi_t^\xi(x) \tag{B.1.2}$$

and, in particular,

$$\xi(x) = \left.\frac{\partial}{\partial t}\varphi_t^\xi(x)\right|_{t=0}, \tag{B.1.3}$$

which could be taken as the definition of the vector field ξ given the mapping φ_t^ξ.

Definition B.1.11. By a ***pseudo-Riemannian manifold*** we shall mean a pair (M,g), where M is a manifold and g is a (non-degenerate) metric tensor on M of signature (p,q), $p+q = m := \dim M$. We shall say that a pseudo-Riemannian manifold is ***Riemannian*** [***Lorentzian***] if $p = m$ (and $q = 0$) [$p = 1$ (and $q = m-1$)].

Definition B.1.12. Let (M, g) be a (pseudo-) Riemannian manifold. A diffeomorphism $\varphi\colon M \to M$ is called an ***isometry*** [a ***conformal isometry***] if $\varphi^* g = g$ [$\varphi^* g = \Omega^2 g$, $\Omega \in C^\infty(M; \mathbb{R}^+)$].

Definition B.1.13. Let (M, g) be a (pseudo-) Riemannian manifold. A vector field ξ on M generating a one-parameter family of isometries [conformal isometries] is called a ***Killing vector field*** [***conformal Killing vector field***].

It is easy to see that the condition for a vector field ξ to be Killing translates into the *Killing equation*

$$\pounds_\xi g = 0$$

(*cf.* §2.1), which, if '∇' denotes the covariant derivative operator associated with the Levi-Civita connection (*cf.* §1.5.1), can be locally written as

$$\pounds_\xi g_{\mu\nu} \equiv \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0, \tag{B.1.4}$$

where $g = g_{\mu\nu}\, \mathrm{d}x^\mu \vee \mathrm{d}x^\nu$, $\xi = \xi^\mu \partial_\mu$ and $\xi_\mu := \xi^\nu g_{\nu\mu}$ in some natural chart. Similarly, the condition for a vector field ξ to be conformal Killing has the following local expression:

$$\pounds_\xi g_{\mu\nu} = \frac{2}{m} \nabla_\rho \xi^\rho g_{\mu\nu}, \tag{B.1.5}$$

$m \equiv \dim M$.

B.2 A simple proposition

We shall now give a direct proof of a simple proposition, which will prove useful in Chapter 1. For its generalization see, e.g., Theorem 3.16 of Kolář *et al.* (1993).

Proposition B.2.1. *Let ξ and η be two vector fields on M. Then,*

$$\tfrac{\partial}{\partial t}(\varphi^\eta_{-t} \circ \varphi^\xi_{-t} \circ \varphi^\eta_t \circ \varphi^\xi_t)\Big|_{t=0} = 0, \tag{B.2.1a}$$

$$\tfrac{1}{2}\tfrac{\partial^2}{\partial t^2}(\varphi^\eta_{-t} \circ \varphi^\xi_{-t} \circ \varphi^\eta_t \circ \varphi^\xi_t)\Big|_{t=0} = [\xi, \eta] \equiv \tfrac{\partial}{\partial t}\varphi^{[\xi,\eta]}_t\Big|_{t=0}. \tag{B.2.1b}$$

Proof. By virtue of (B.1.1b) and (B.1.3) the first order expansion of $\varphi^\xi_t(x)$ around $t = 0$ reads

$$\varphi^\xi_t(x) = x + t\xi(x) + O(t^2), \tag{B.2.2}$$

and analogously for $\varphi^\eta_t(x)$. Hence [omitting the points (of M) at which maps are evaluated], we find

$$\frac{\partial}{\partial t}(\varphi^\eta_{-t} \circ \varphi^\xi_{-t} \circ \varphi^\eta_t \circ \varphi^\xi_t)\bigg|_{t=0} = \frac{\partial}{\partial t}\Big(\mathrm{id}_M + O(t^2)\Big)\bigg|_{t=0} = 0,$$

which is nothing but (B.2.1a). To prove (B.2.1b), we need a second order expansion of $\varphi^\xi_t(x)$ and $\varphi^\eta_t(x)$ since we need to compute a second derivative here, all higher order terms again vanishing owing to the fact that the expression is evaluated at $t = 0$. Taking the derivative of (B.1.2), applying the chain rule and using (B.1.3), we then get

$$\left.\frac{\partial^2}{\partial t^2}\varphi_t(x)\right|_{t=0} = T_{\varphi_t(x)}\xi \circ \xi(x). \tag{B.2.3}$$

Therefore, by virtue of (B.2.2) and (B.2.3) the second order expansion of $\varphi_t(x)$ around $t = 0$ reads

$$\varphi_t(x) = x + t\xi(x) + t^2 T_{\varphi_t(x)}\xi \circ \xi(x) + O(t^3),$$

and analogously for $\varphi_t^\eta(x)$. Hence, we readily find

$$\left.\frac{1}{2}\frac{\partial^2}{\partial t^2}(\varphi_{-t}^\eta \circ \varphi_{-t}^\xi \circ \varphi_t^\eta \circ \varphi_t^\xi)\right|_{t=0} = \left.\frac{1}{2}\frac{\partial^2}{\partial t^2}\Big(\mathrm{id}_M + t^2[\xi,\eta] + O(t^3)\Big)\right|_{t=0} = [\xi,\eta],$$

as claimed. □

Appendix C

Lie groups and Lie algebras

In this appendix we collect together some standard results on Lie groups, Lie algebras and Lie group actions on manifolds. The reader is referred to Kolář *et al.* (1993), §§4–5, for a concise, yet entirely self-contained, introduction to the same concepts, and for all the proofs of the results presented herein.

C.1 Lie groups

Definition C.1.1. A ***Lie group*** G is a (smooth) manifold and a group such that the multiplication $\mu\colon G \times G \to G$ is smooth.

Then, it can be shown that also the inversion $\iota\colon G \to G$ is smooth. Throughout, we shall use a multiplicative notation, whereby multiplication in G is denoted by juxtaposition, i.e. $\mu(a, b) =: ab$, and $\iota(a) =: a^{-1}$ for all $a, b \in G$. Furthermore, we shall denote by e the identity of G. By the *dimension* of G we shall mean the dimension of the underlying (finite-dimensional) manifold. By the *left translation* L_a by an element $a \in G$ we shall mean the diffeomorphism $L_a\colon G \to G$ such that $L_a\colon b \mapsto L_a b := ab$ for all $b \in G$. Analogously, by the *right translation* R_a by an element $a \in G$ we shall mean the diffeomorphism $R_a\colon G \to G$ such that $R_a\colon b \mapsto R_a b := ba$ for all $b \in G$. It follows immediately that $L_a \circ L_b = L_{ab}$, $R_a \circ R_b = R_{ba}$, $L_a^{-1} = L_{a^{-1}}$, $R_a^{-1} = R_{a^{-1}}$ and $L_a \circ R_b = R_b \circ L_a$.

Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$. Examples of Lie groups we shall encounter in this thesis are the *general linear group* $\mathrm{GL}(\mathcal{V})$ of all automorphisms (i.e. invertible linear mappings) of a finite-dimensional vector space $\mathcal{V}$ into itself [in particular, we shall set $\mathrm{GL}(m, \mathbb{K}) := \mathrm{GL}(\mathbb{K}^m)$], the *special linear group* $\mathrm{SL}(m, \mathbb{K})$ of all $m \times m$ invertible matrices with unit determinant, the *pseudo-orthogonal group* $\mathrm{O}(p, q)$ of all linear isometries of $(\mathbb{R}^m, \eta)$, where η is the standard *Minkowski metric* of signature (p, q), $p + q = m$, on $\mathbb{R}^m$, i.e.

$$\|\eta_{ab}\| := \mathrm{diag}(\underbrace{1, \ldots, 1}_{p \text{ times}}, \underbrace{-1, \ldots, -1}_{q \text{ times}}).$$

We shall also denote by $\mathrm{SO}(p, q)$ the *special pseudo-orthogonal group* formed by all elements of $\mathrm{O}(p, q)$ with unit determinant, and by $\mathrm{SO}(p, q)^e$ its connected component with the identity. Finally, we shall also need the *pseudo-conformal group* $\mathrm{CSO}(p, q) :=$

$\mathrm{SO}(p,q) \times \mathbb{R}^+$ of all conformal isometries of $(\mathbb{R}^m, \eta)$, as well as the *general affine group* $\mathrm{GA}(m,\mathbb{K}) := \mathrm{GL}(m,\mathbb{K}) \rtimes \mathbb{K}^m$ formed by all matrices $\tilde{\mathsf{A}}$ of the form

$$\tilde{\mathsf{A}} = \begin{pmatrix} \mathsf{A} & v \\ 0 & 1 \end{pmatrix},$$

$\mathsf{A} \in \mathrm{GL}(m,\mathbb{K})$, $v \in \mathbb{K}^m$.

Definition C.1.2. Let H be a Lie group. A subgroup G of H is called a ***Lie subgroup*** if G itself is a Lie group and the inclusion $i\colon G \to H$ is smooth.

Proposition C.1.3. *Let G be a closed subgroup of a Lie group H. Then, G is a Lie subgroup and a submanifold of H.*

Thus, for instance, $\mathrm{SO}(p,q)$ is a Lie subgroup of $\mathrm{SL}(m,\mathbb{R})$, which is in turn a Lie subgroup of $\mathrm{GL}(m,\mathbb{R})$.

Definition C.1.4. By the ***centre*** of a Lie group G we shall mean the Lie subgroup C of G formed by all elements $a \in G$ such that $ab = ba$ for all $b \in G$.

Definition C.1.5. By a ***Lie group homomorphism*** between two Lie groups G and H we shall mean a smooth mapping $\varphi\colon G \to H$ such that $\varphi(ab) = \varphi(a)\varphi(b)$ for all $a, b \in G$.

As usual, we shall speak of *Lie group isomorphism*, *endomorphism* or *automorphism*, if φ is invertible, $H = G$, or φ is an invertible endomorphism, respectively. For $a \in G$, define $I_a\colon G \to G$ by $I_a b := aba^{-1} \equiv (L_a \circ R_{a^{-1}})b \equiv (R_{a^{-1}} \circ L_a)b$. Then, I_a is clearly an automorphism of G, called the *inner automorphism* (or *conjugation*) by a.

Definition C.1.6. Let $(\mathbb{R}, +)$ denote the (Lie) group of real numbers with addition. By a ***one-parameter subgroup*** of a Lie group G we shall mean a Lie group homomorphism $\varphi\colon (\mathbb{R}, +) \to G$, i.e. a smooth curve φ in G such that $\varphi(0) = e$ and $\varphi(s+t) = \varphi(s)\varphi(t)$ (*cf.* §B.1).

Definition C.1.7. By a ***real*** [***complex***] ***representation of a Lie group*** G on a real [complex] (finite-dimensional) vector space $\mathcal{V}$ we shall mean a Lie group homomorphism $\rho\colon G \to \mathrm{GL}(\mathcal{V})$. A representation is said to be *faithful* if it is injective.

C.2 Lie algebras

Definition C.2.1. Let $\mathcal{V}$ be a vector space on a field $\mathbb{K} = \mathbb{R}, \mathbb{C}$ endowed with a binary operation

$$\begin{cases} [\cdot,\cdot]\colon \mathcal{V} \times \mathcal{V} \to \mathcal{V} \\ [\cdot,\cdot]\colon (v, v') \mapsto [v, v'] \end{cases},$$

called the *commutator* (or *Lie bracket*) of v and v', satisfying the following properties:

(*i*) $[v, \alpha v' + \beta v''] = \alpha[v, v'] + \beta[v, v'']$ for all $\alpha, \beta \in \mathbb{K}$ and $v, v', v'' \in \mathcal{V}$,

(*ii*) $[v, v'] = -[v', v]$, for all $v, v' \in \mathcal{V}$, *(iii)* $[v, [v', v'']] + [v', [v'', v]] + [v'', [v, v']] = 0$, for all $v, v', v'' \in \mathcal{V}$.

Then, the pair $(\mathcal{V}, [\cdot, \cdot])$ or, for short, simply $\mathcal{V}$ is called a ***Lie algebra*** (on $\mathbb{K}$).

Properties *(i)* and *(ii)* simply express the *linearity* and *antisymmetry* of the commutator[1], whereas *(iii)* is known as the *Jacobi identity.* As a first example of a Lie algebra, recall the (infinite-dimensional) Lie algebra $\mathfrak{X}(M)$ of all vector fields on a manifold M encountered in Definition B.1.3. By the *dimension* of a Lie algebra we shall mean the dimension of the underlying (finite-dimensional) vector space.

Definition C.2.2. By a ***Lie subalgebra*** of a Lie algebra $\mathcal{V}$ we shall mean a vector subspace $\mathcal{S}$ of $\mathcal{W}$ such that $[v, v'] \in \mathcal{S}$ for all $v, v' \in \mathcal{S}$.

Definition C.2.3. By an ***ideal*** of a Lie algebra $\mathcal{V}$ we shall mean a Lie subalgebra $\mathcal{I}$ of $\mathcal{V}$ such that $[v, v'] \in \mathcal{I}$ for all $v \in \mathcal{V}$, $v' \in \mathcal{I}$.

Definition C.2.4. By the ***centre*** of a Lie algebra $\mathcal{V}$ we shall mean the ideal $\mathcal{C}$ of $\mathcal{V}$ formed by all elements $v' \in \mathcal{V}$ such that $[v, v'] = 0$ for all $v \in \mathcal{V}$.

Definition C.2.5. By a ***Lie algebra homomorphism*** between two Lie algebras $\mathcal{V}$ and $\mathcal{W}$ we shall mean a linear mapping $\varphi\colon \mathcal{V} \to \mathcal{W}$ such that $\varphi([v, v']) = [\varphi(v), \varphi(v')]$ for all $v, v' \in \mathcal{V}$.

As an example, it easy to show that the push-forward φ_* by a diffeomorphism $\varphi\colon M \to N$ is a Lie algebra homomorphism $\mathfrak{X}(M) \to \mathfrak{X}(N)$.

Now, a vector field ξ on G is called *left-invariant* if $(L_a)_*\xi = \xi$ for all $a \in G$. Since the push-forward is a Lie algebra homomorphism, the space $\mathfrak{X}_{\mathrm{L}}(G)$ of all left invariant vector field on G is closed under the Lie bracket, and hence it is a Lie subalgebra of $\mathfrak{X}(G)$. Furthermore, any left-invariant vector field is uniquely determined by its value $\xi(e) \in T_eG$ at the identity. Indeed, $\xi(a) = (L_{a^{-1}})_*\xi(a) = T_eL_a\xi(e)$ for all $a \in G$. Thus, $\mathfrak{X}_{\mathrm{L}}(G)$ is linearly isomorphic to T_eG, and the commutator on $\mathfrak{X}_{\mathrm{L}}(G)$ induces a Lie algebra structure on T_eG, which is known as the ***Lie algebra of the Lie group*** G and will be denoted by $\mathfrak{g}$.

Analogously, a vector field ξ on G is called *right-invariant* if $(R_a)_*\xi = \xi$ for all $a \in G$. Right-invariant vector fields form a Lie subalgebra $\mathfrak{X}_{\mathrm{R}}(G)$ of $\mathfrak{X}(G)$, which is again isomorphic to $\mathfrak{g}$. In particular, we can define a basis $(\rho_{\mathcal{A}})$ of right-invariant vector fields on G by

$$\rho_{\mathcal{A}}(a) = T_eR_a\varepsilon_{\mathcal{A}}$$

for all $a \in G$, $(\varepsilon_{\mathcal{A}})_{\mathcal{A}=1}^{\dim \mathfrak{g}}$ being a basis of T_eG. Indeed, since $R_a\colon G \to G$ is a diffeomorphism for all $a \in G$, $T_eR_a\colon T_eG \to T_aG$ must be a linear isomorphism for all $a \in G$, so that $(\rho_{\mathcal{A}}(a))$ is a basis of T_aG for all $a \in G$. It remains to show that the $\rho_{\mathcal{A}}$'s are right-invariant. We have

$$\begin{aligned}
(R_a)_*\rho_{\mathcal{A}}(b) &= T_{ba^{-1}}R_a \circ \rho_{\mathcal{A}} \circ R_{a^{-1}}(b) \\
&= (T_{ba^{-1}}R_a \circ T_eR_{ba^{-1}})\varepsilon_{\mathcal{A}} \\
&= T_e(R_a \circ R_{ba^{-1}})\varepsilon_{\mathcal{A}} \\
&= T_eR_b\varepsilon_{\mathcal{A}} = \rho_{\mathcal{A}}(b)
\end{aligned}$$

[1]Note that *(i)* and *(ii)* imply linearity also in the first argument.

for all $a, b \in G$, demonstrating that the $\rho_{\mathcal{A}}$'s are indeed right-invariant. □

Now, let G be a group of matrices, e.g. $\mathrm{GL}(m, \mathbb{R})$ or any of its Lie subgroups, and let $\varphi\colon (\mathbb{R}, +) \to G$ a one-parameter subgroup of G generated by a left-invariant vector field ξ. Then, we have (*cf.* §B.1)

$$\begin{aligned} \tfrac{\partial\varphi}{\partial t} &= \xi(\varphi(t)) \\ &= T_e L_{\varphi(t)} \xi_e \\ &= \varphi(t)\xi_e, \end{aligned}$$

where $\xi_e := \xi(e) \in T_e G$. This equation has a unique solution satisfying the initial condition $\varphi(0) = e$, which is

$$\varphi(t) = \exp(t\xi_e),$$

exp denoting the exponential mapping of matrices given by the well-known expression $\exp(t\mathsf{A}) = \sum_{k=0}^{\infty} t^k/k!\, \mathsf{A}^k$ for all matrix A. This argument motivates the following

Definition C.2.6. Let G be a Lie group and $\mathfrak{g}$ its Lie algebra. The mapping $\exp\colon \mathfrak{g} \to G$ defined by $\exp \xi_e := \varphi(1)$, where φ is the one-parameter subgroup of G with $\partial\varphi/\partial t|_{t=0} = \xi_e$, is called the ***exponential mapping***.

Proposition C.2.7. *The exponential mapping is a diffeomorphism from a neighbourhood of* $0 \in \mathfrak{g}$ *to a neighbourhood of* $e \in G$. *If* G *is connected and* U *is an open subset of* $\mathfrak{g}$ *containing* 0, *then* $\exp U$ *equals* G. *If* G *is not connected, then* $\exp U$ *equals the connected component* G^e *of* G *containing* e.

Thanks to the exponential mapping, we can now easily find the relationship between the Lie groups listed in the previous section and their respective Lie algebras: as a rule, we shall denote a Lie algebra with the lower-case gothic version of the letter combination denoting the corresponding Lie group.

As an exercise, we shall compute the Lie algebra $\mathfrak{so}(p, q)$ of $\mathrm{SO}(p, q)$, which, by Proposition C.2.7, is the same as the Lie algebra of $\mathrm{O}(p, q)$ or $\mathrm{SO}(p, q)^e$. Now, an element $\mathsf{O} \in \mathrm{SO}(p, q)$ is characterized by the condition $\mathsf{O}^\top = \mathsf{O}^{-1}$, where $\mathsf{O}^\top$ denotes the adjoint ("transpose") of O with respect to η defined by requiring $\eta(\mathsf{O}^\top v, v') = \eta(v, \mathsf{O}v')$ for all $v, v' \in \mathbb{R}^m$. Then, using the exponential mapping, we certainly have

$$\exp 0 = \mathsf{I} = (\exp \mathsf{A})^\top \exp \mathsf{A} = \exp(\mathsf{A}^\top + \mathsf{A})$$

for some $\mathsf{A} \in \mathfrak{so}(p, q)$ such that $\exp \mathsf{A} = \mathsf{O}$, I denoting the identity of $\mathrm{SO}(p, q)$. Hence, $\mathsf{A}^\top = -\mathsf{A}$, and we deduce that $\mathfrak{so}(p, q)$ is the Lie algebra formed by all antisymmetric (real) matrices. Analogously, we deduce that $\mathfrak{sl}(m, \mathbb{K})$ is formed by all traceless (real or complex) matrices, that $\mathfrak{cso}(p, q) = \mathfrak{so}(p, q) \oplus \mathbb{R}$, and $\mathfrak{ga}(m, \mathbb{K}) = \mathfrak{gl}(m, \mathbb{K}) \oplus \mathbb{K}^m$ (semi-direct sum), $\mathfrak{gl}(m, \mathbb{K})$ denoting the Lie algebra of all endomorphisms of $\mathbb{K}^m$.

Definition C.2.8. By a ***real*** **[*complex*]** ***representation of a Lie algebra*** $\mathcal{W}$ on a real [complex] (finite-dimensional) vector space $\mathcal{V}$ we shall mean a Lie algebra homomorphism $\rho'\colon \mathcal{W} \to \mathfrak{gl}(\mathcal{V})$, $\mathfrak{gl}(\mathcal{V})$ denoting the Lie algebra of all endomorphism of $\mathcal{V}$. A representation is said to be *faithful* if it is injective.

Proposition C.2.9. *Let $\varphi\colon G \to G'$ be a Lie group homomorphism, and let $\mathfrak{g}$ and $\mathfrak{g}'$ denote the Lie algebras of G and G', respectively. Then, $T_e\varphi\colon \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism.*

Now, for $a \in G$ define the mapping $\mathrm{Ad}_a\colon \mathfrak{g} \to \mathfrak{g}$ as $\mathrm{Ad}_a := T_e I_a$, which turns out to be a Lie algebra homomorphism by virtue of the previous proposition. Moreover, $\mathrm{Ad}\colon G \to \mathrm{GL}(\mathfrak{g})$ given by $\mathrm{Ad}(a) := \mathrm{Ad}_a$ is a representation of G, called the ***adjoint representation of*** G. Finally, we define the ***adjoint representation*** $\mathrm{ad}\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ ***of*** $\mathfrak{g}$ as $\mathrm{ad} := T_e\mathrm{Ad}$. We then have the following important proposition, the proof of which is straightforward for a matrix group.

Proposition C.2.10. *It holds*

$$\mathrm{ad}_{\xi_e}\eta_e := \mathrm{ad}(\xi_e) \circ \eta_e = [\xi_e, \eta_e]$$

for all $\xi_e, \eta_e \in \mathfrak{g}$.

C.3 Lie group actions on manifolds

Definition C.3.1. A ***left*** [***right***] ***action*** of a Lie group G on a manifold M is a smooth mapping $\tilde{L}\colon G \times M \to M$ [$\tilde{R}\colon M \times G \to M$] such that $\tilde{L}_a \circ \tilde{L}_b = \tilde{L}_{ab}$ [$\tilde{R}_a \circ \tilde{R}_b = \tilde{R}_{ba}$] and $\tilde{L}_e = \mathrm{id}_M$ [$\tilde{R}_e = \mathrm{id}_M$], where $\tilde{L}_a x := \tilde{L}(a,x) =: a \cdot x$ [$\tilde{R}_a x := \tilde{R}(x,a) =: x \cdot a$] for all $a \in G$, $x \in M$.

A *G-manifold* is a manifold M together with a left or right action of a Lie group G on M. Given a left [right] action of a Lie group G on M, by the *orbit* through a point $x \in M$ we shall mean the set $G \cdot x := \tilde{L}(G,x) \equiv \{\, x' \in M \mid \exists a \in G : x' = a \cdot x \,\} \subset M$ [$x \cdot G := \tilde{R}(x,G) \equiv \{\, x' \in M \mid \exists a \in G : x' = x \cdot a \,\} \subset M$]. Of course, an orbit is an equivalence class of points in M. Accordingly, the *space of orbits* of M will be denoted by M/G.

The action is called *transitive* if M is one orbit, i.e. for all $x, x' \in M$ there is some $a \in G$ such that $a \cdot x = x'$ [$x \cdot a = x'$]. The action is called *free* if $a \cdot x = a' \cdot x$ [$x \cdot a = x \cdot a'$] for some $x \in M$ implies $a = a'$. Finally, the action is called *effective* if $\tilde{L}_a = \tilde{L}_{a'}$ [$\tilde{R}_a = \tilde{R}_{a'}$] implies $a = a'$.

Appendix D

Clifford algebras and spinors

This appendix consists of a summary of the basic concepts of the theory of spinors.

D.1 Clifford algebra, γ matrices and spin group

A ***Clifford algebra*** $\mathcal{C}\ell(\mathcal{V})$ on a (real) m-dimensional vector space $\mathcal{V}$ equipped with a scalar product $(v, v') \mapsto g(v, v')$ of signature (p, q), $p + q = m$, is an associative algebra such that there exists a linear map γ from $\mathcal{V}$ into a subset of $\mathcal{C}\ell(\mathcal{V})$ generating $\mathcal{C}\ell(\mathcal{V})$ and satisfying the property

$$\gamma(v)\gamma(v') + \gamma(v')\gamma(v) = -2g(v, v')e,$$

e denoting the unit element of $\mathcal{C}\ell(\mathcal{V})$. The Clifford algebra on an m-dimensional vector space has dimension 2^m. It can be realized by an algebra of linear maps of a complex vector space of dimension $2^{I(m/2)}$ into itself, $I(m/2)$ denoting the integral part of $m/2$. Henceforth, we shall always assume that m is even.

By γ *matrices* we shall mean a set of m such linear maps, represented by matrices, associated with the vectors of an orthonormal frame of $\mathcal{V}$.

If we denote by (η_{ab}) the components of g in such a frame (*cf.* §C.1), then the γ matrices, which we shall denote by (γ_a), satisfy the fundamental relation

$$\gamma_a\gamma_b + \gamma_b\gamma_a = -2\eta_{ab}, \tag{D.1.1}$$

where the identity matrix is implied on the right-hand side. We shall also define

$$\gamma_{a_1 \ldots a_k} := \frac{1}{k!}\gamma_{[a_1} \cdots \gamma_{a_k]}. \tag{D.1.2}$$

In fact, it turns out that we need to consider only antisymmetrized products. This is because, on applying (D.1.1) iteratively, we find

$$\gamma_a\gamma_b = \gamma_{ab} - \eta_{ab},$$
$$\gamma_a\gamma_b\gamma_c = \gamma_{abc} - \eta_{ab}\gamma_c - \eta_{bc}\gamma_a + \eta_{ca}\gamma_b,$$

and so forth. Moreover, substituting the former into the latter relation yields

$$\gamma_{ab}\gamma_c + \gamma_c\gamma_{ab} = 2\gamma_{abc}, \tag{D.1.3}$$

an identity which proves useful in Chapter 4. Furthermore, it can be shown that the γ matrices satisfy the following properties:

$$\gamma_a^\dagger = \begin{cases} -\gamma_a & \text{if } a \in \{0,\ldots,p\}, \\ \gamma_a & \text{if } a \in \{p+1,\ldots,p+q \equiv m\}, \end{cases} \tag{D.1.4}$$

'†' denoting "transposition" (i.e. adjunction by g) and complex conjugation. From (D.1.1) and (D.1.4) it follows immediately that, in the particular case $(p = 1, q = m - 1)$,

$$\gamma_0\gamma_a\gamma_0^{-1} = -\gamma_a^\dagger$$

for all $a \in \{1,\ldots,m\}$.

Finally, by the *spin group* $\mathrm{Spin}(p,q)$ we shall mean the (Lie) subgroup of $\mathrm{GL}(m,\mathbb{C})$ consisting of those elements S such that there exists an $L \in \mathrm{SO}(p,q)$ satisfying

$$S\gamma_a S^{-1} = L_a{}^b\gamma_b, \tag{D.1.5a}$$

$L \equiv \|L_a{}^b\|$, and such that

$$\det(S) = 1. \tag{D.1.5b}$$

Relations (D.1.5) define an epimorphism from $\mathrm{Spin}(p,q)$ onto $\mathrm{SO}(p,q)$. It can be shown that $\mathrm{Spin}(p,q)$ $[\mathrm{Spin}(p,q)^e]$ is the twofold covering of $\mathrm{SO}(p,q)$ $[\mathrm{SO}(p,q)^e]$, the superscript e denoting the connected component with the unit. In particular, $\mathrm{Spin}(p,q)^e$ is simply connected.

The Lie group epimorphism $\Lambda\colon \mathrm{Spin}(p,q)^e \to \mathrm{SO}(p,q)^e$ induces a Lie algebra isomorphism $\Lambda' := T_e\Lambda\colon \mathfrak{spin}(p,q) \to \mathfrak{so}(p,q)$. On differentiating (D.1.5a) and taking (D.1.5b) into account, we find that $(\Lambda')^{-1}(\mathsf{A})$ is given by

$$(\Lambda')^{-1}(\mathsf{A}) \equiv -\frac{1}{4}\mathsf{A}^{ab}\gamma_{ab} \tag{D.1.6}$$

for all $\mathsf{A} \in \mathfrak{so}(p,q)$, $\mathsf{A} \equiv \|\mathsf{A}^a{}_c =: \mathsf{A}^{ab}\eta_{bc}\|$.

D.2 Spin structures and spinors

Definition D.2.1. Let (M,g) denote a (pseudo-) Riemannian manifold, i.e. a manifold M equipped with a metric tensor g of signature (p,q) (*cf.* §B.1). A ***spin structure*** on (M,g) is a pair $(\mathrm{Spin}(M,g),\tilde{\Lambda})$, where

(*i*) $\mathrm{Spin}(M,g)$ is principal bundle over M with structure group $\mathrm{Spin}(p,q)^e$ called the *spin bundle*;

(*ii*) $\tilde{\Lambda}\colon \mathrm{Spin}(M,g) \to \mathrm{SO}(M,g)$ is a principal morphism with respect to the Lie group epimorphism $\Lambda\colon \mathrm{Spin}(p,q)^e \to \mathrm{SO}(p,q)^e$, i.e. the following diagrams are commutative:

$$\begin{array}{ccc} \mathrm{Spin}(M,g) & \xrightarrow{\tilde{\Lambda}} & \mathrm{SO}(M,g) \\ & \searrow \quad \swarrow & \\ & M & \end{array} \qquad \begin{array}{ccc} \mathrm{Spin}(M,g) & \xrightarrow{\tilde{R}_S} & \mathrm{Spin}(M,g) \\ \tilde{\Lambda}\downarrow & & \downarrow\tilde{\Lambda} \\ \mathrm{SO}(M,g) & \xrightarrow[\tilde{R}_{\Lambda(S)}]{} & \mathrm{SO}(M,g) \end{array}$$

Note that there are topological obstructions for a (pseudo-) Riemannian manifold (M, g) to be a *spin manifold*, i.e. for (M, g) to admit a spin structure (*cf.*, e.g., Lawson & Michelsohn 1989). In the rest of this thesis, we shall tacitly assume that all manifolds under consideration satisfy all the conditions required for this to be possible. Indeed, for $m = 4$ these conditions are satisfied by all physically admissible space-times.

Also, in Chapter 4 we shall give a slightly different definition of a spin structure, which avoids any reference to a metric g on M, and is hence more suitable for general relativity, where the metric is supposed not to be given *a priori*, but to be determined *a posteriori* by Einstein's field equations.

Definition D.2.2. Let $\hat{\gamma}$ be the linear representation of $\mathrm{Spin}(p,q)^e$ on the vector space $\mathbb{C}^m$, $p+q=m$, induced by a given choice of γ matrices. By a ***spinor field*** (or, more precisely, a ***spin-vector field***) we shall mean a section ψ of the associated vector bundle $\mathrm{Spin}(M,g)_{\hat{\gamma}} := \mathrm{Spin}(M,g) \times_{\hat{\gamma}} \mathbb{C}^m$.

Now, let $\mathsf{h}\colon \mathbb{C}^m \times \mathbb{C}^m \to \mathbb{C}$ be the $\hat{\gamma}$-invariant scalar product on $\mathbb{C}^m$ given by $\mathsf{h}(\hat{\gamma}(S)v, \hat{\gamma}(S)v') = \mathsf{h}(v, v')$ for all $S \in \mathrm{Spin}(p,q)^e$, $v, v' \in \mathbb{C}^m$. Denote by $v \mapsto \tilde{v}$ the anti-isomorphism $\mathbb{C}^m \to (\bar{\mathbb{C}}^m)^*$ induced by h defined by $\tilde{v}v' = \mathsf{h}(v, v')$. If ψ is a spin-vector field, the spin-covector field $\tilde{\psi}$ given by $\tilde{\psi}(x) = \widetilde{\psi(x)}$ for all $x \in M$ is called the *Dirac adjoint* of ψ. In the case $(p = 1, q = m-1)$, $\tilde{\psi}(x)$ locally reads

$$\tilde{\psi}(x) = \psi^\dagger(x)\gamma_0, \tag{D.2.1}$$

where, as usual, $\psi^\dagger(x)$ stands for $(\psi(x))^\dagger$.

We can also define the *spinor connection* $\tilde{\omega}$, understood as a $\mathfrak{spin}(p,q)$-valued 1-form on $\mathrm{Spin}(M,g)$, corresponding to a given principal connection ω on $\mathrm{SO}(M,g)$, understood as a $\mathfrak{so}(p,q)$-valued 1-form on $\mathrm{SO}(M,g)$ (*cf.* §1.5), as

$$\tilde{\omega} := (\Lambda')^{-1} \circ \tilde{\Lambda}^*\omega. \tag{D.2.2}$$

By virtue of (D.1.6), the components $(\tilde{\omega}_\mu)$ of $\tilde{\omega}$ then read

$$\tilde{\omega}_\mu \equiv -\frac{1}{4}\omega^{ab}{}_\mu\gamma_{ab}, \tag{D.2.3}$$

$(\omega^a{}_{c\mu} =: \omega^{ab}{}_\mu\eta_{bc})$ denoting the components of ω (*cf.* §1.5.1). From identity (D.2.3) it follows that the local expressions of the covariant derivative of a spinor and its Dirac adjoint are

$$\nabla_\mu \psi = \partial_\mu \psi - \frac{1}{4}\omega^{ab}{}_\mu \gamma_{ab}\psi, \qquad \text{(D.2.4)}$$
$$\nabla_\mu \tilde{\psi} = \widetilde{\nabla_\mu \psi} \equiv \partial_\mu \tilde{\psi} + \frac{1}{4}\tilde{\psi}\omega^{ab}{}_\mu \gamma_{ab}.$$

D.3 2-spinors

Throughout this section, M will denote a 4-dimensional spin manifold. Now, in the case $m = 4$, we have the notable Lie group isomorphism $\mathrm{Spin}(1,3)^e \cong \mathrm{SL}(2,\mathbb{C})$, and the corresponding Lie algebra isomorphism $\mathfrak{spin}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$. Therefore, the following definition makes sense.

Definition D.3.1. Let λ be the standard action of $\mathrm{SL}(2,\mathbb{C})$ on the vector space $\mathbb{C}^2$. By a ***2-spinor field*** (or, more precisely, a ***2-spin-vector field***) we shall mean a section ϕ of the associated vector bundle $S(M,g) := \mathrm{Spin}(M,g) \times_\lambda \mathbb{C}^2$.

From $S(M,g)$ we can construct in a canonical fashion three more bundles. Indeed, let $S(M,g)^{\mathbb{C}} := S(M,g) \otimes \mathbb{C}$ be the complexification of $S(M,g)$. On $S(M,g)^{\mathbb{C}}$ there is a conjugation defined, and we let $\bar{S}(M,g)$ be the conjugate of $S(M,g)$ in $S(M,g)^{\mathbb{C}}$. Finally, we let $S^*(M,g)$ and $\bar{S}^*(M,g)$ denote the (complex-linear) duals of the vector bundles $S(M,g)$ and $\bar{S}(M,g)$, respectively.

A (4-) spinor field ψ can be then represented by a pair (ϕ,ϕ') formed by 2-spinor fields ϕ and ϕ', sections of $S(M,g)$ and $\bar{S}(M,g)$, respectively. We refer to Penrose & Rindler (1984, 1986) for more detail. Here, we shall just mention the notable vector bundle isomorphisms $TM^{\mathbb{C}} := TM \otimes \mathbb{C} \cong S(M,g) \otimes \bar{S}(M,g)$ and $TM \cong \mathrm{Re}(S(M,g) \otimes \bar{S}(M,g))$, induced by the group epimorphism $\Lambda\colon \mathrm{Spin}(1,3)^e \cong \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}(1,3)^e$ and the isomorphism $\mathbb{C}^4 \cong \mathbb{C}^2 \oplus \bar{\mathbb{C}}^2$. In local anholonomic coordinates, we can represent these vector bundle isomorphisms by means of the (globally invariant, hermitian) *Infeld-van der Waerden symbols* $(\sigma^a{}_{AA'})$ as $v^a = \sigma^a{}_{AA'}v^{AA'}$ (*cf.* §1.1), or simply as $v^a = v^{AA'}$ with the standard identification $a \leftrightarrow AA'$, where $\bar{v}^{AA'} := \overline{v^{AA'}} = v^{AA'}$ for the latter isomorphism. Observe that, here, $TM^{\mathbb{C}}$ and TM are tacitly regarded as vector bundles *associated with* $\mathrm{SO}(M,g)$, *not* LM (*cf.* §1.4 and Example 1.10.15): this fact has crucial implications for the theory of Lie derivatives (*cf.* Remark 2.1.5 and §2.7.1).

Finally, note, that owing to its 2-dimensionality, there is—modulo rescalings—only one symplectic (i.e. closed non-degenerate) 2-form ε on $S(M,g)$, whose components will be denoted by (ε_{AB}). We shall also set $\varepsilon_{A'B'} := \overline{\varepsilon_{AB}}$, as customary. The 2-forms ε and $\bar{\varepsilon}$ induce an inner product on $S(M,g)$ and $\bar{S}(M,g)$, respectively, and we shall use the standard conventions

$$\phi_A := \phi^B \varepsilon_{BA} \qquad \text{and} \qquad \varepsilon^{AC}\varepsilon_{BC} = \varepsilon_B{}^A = \delta^A{}_B.$$

With these conventions, Clifford equation (D.1.1) can be recast into the form

$$\sigma_a{}^A{}_{A'}\sigma_b{}^{A'}{}_B + \sigma_b{}^A{}_{A'}\sigma_a{}^{A'}{}_B = -\varepsilon_B{}^A \eta_{ab}.$$

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